1 Simple Harmonic Motion

1.1 PERIODIC MOTION

When a body repeats its path of motion back and forth about the equilibrium or mean position, the motion is said to be periodic. All periodic motions need not be back and forth like the motion of the earth about the sun, which is periodic but not vibratory in nature.

1.2 THE TIME PERIOD (T)

The time period of a vibrating or oscillatory system is the time required to complete *one full cycle* of vibration of oscillation.

1.3 THE FREQUENCY (ν)

The frequency is the number of complete oscillations or cycles per unit time. If T is the time for one complete oscillation.

$$\nu = \frac{1}{T} \quad ...(1.1)$$

1.4 THE DISPLACEMENT (*X* OR *Y*)

The displacement of a vibrating body is the distance from its equilibrium or mean position. The maximum displacement is called the *amplitude*.

1.5 RESTORING FORCE OR RETURN FORCE

The mass m lies on a frictionless horizontal surface. It is connected to one end of a spring of negligible mass and relaxed length a_0, whose other end is fixed to a rigid wall W [Fig. 1.1 (a)].

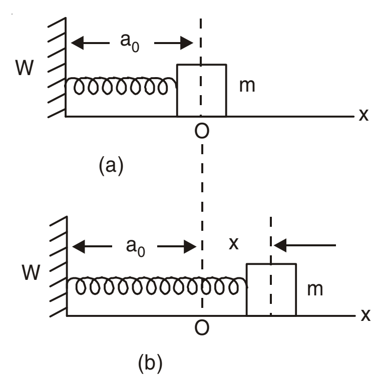

Fig 1.1

If the mass m is given a displacement along the x-axis and released [Fig. 1.1 (b)], it will oscillate back and forth in a straight line along x-axis about the equilibrium position O. Suppose at any instant of time the displacement of the mass is x from the equilibrium position. There is a force tending to restore m to its equilibrium position. This force, called the *restoring force* or return force, is proportional to the displacement x when x is not large:

$$\vec{F} = -k\,x\,\hat{i} \qquad ...(1.2)$$

where k, the constant of proportionality, is called the *spring constant* or *stiffness factor,* and $\hat{i}$ is the unit vector in the positive x-direction. The minus sign indicates that the restoring force is always opposite in direction to the displacement.

By Newton's second law Eqn. (1.2) can be written as

$$m\ddot{x} = -kx \text{ or, } \ddot{x} + \omega^2 x = 0 \qquad ...(1.3)$$

where $\omega^2 = k/m$ = return force per unit displacement per unit mass. ω is called the angular frequency of oscillation.

1.6 SIMPLE HARMONIC MOTION (SHM)

If the restoring force of a vibrating or oscillatory system is proportional to the displacement of the body from its equilibrium position and is directed opposite to the direction of displacement, the motion of the system is simple harmonic and it is given by Eqn. (1.3). Let the initial conditions be $x = A$ and $\dot{x} = 0$ at $t = 0$, then integrating Eqn. (1.3), we get

$$x(t) = A\cos\omega t \qquad ...(1.4)$$

where A, the maximum value of the displacement, is called the amplitude of the motion. If T is the time for one complete oscillation, then

$$x(t+T) = x(t)$$

or

$$A\cos\omega(t+T) = A\cos\omega t$$

or

$$\omega T = 2\pi$$

or

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \qquad ...(1.5)$$

and

$$\nu = \frac{1}{T} = \frac{\omega}{2\pi} \quad \text{or,} \quad \omega = 2\pi\nu.$$

The general solution of Eqn. (1.3) is

$$x(t) = C\cos\omega t + D\sin\omega t \qquad ...(1.6)$$

where C and D are determined from the initial conditions. Euqation (1.6) can be written as

$$x(t) = A\cos(\omega t - \phi) \qquad ...(1.7)$$

where $C = A\cos\phi$ and $D = A\sin\phi$. The amplitude for the motion described by Eqn. (1.7) is now $A = (C^2 + D^2)^{1/2}$ and the angular frequency is ω which is uneffected by the initial conditions. The angle ϕ called the *phase angle or phase constant or epoch* is given by $\phi = \tan^{-1}(D/C)$, where ϕ is chosen in the interval $0 \le \phi \le 2\pi$.

1.7 VELOCITY, ACCELERATION AND ENERGY OF A SIMPLE HARMONIC OSCILLATOR

From Eqn. (1.7), we find that the magnitude of the velocity v is

$$v = |-A\,\omega\sin(\omega t - \phi)| = A\omega(1 - x^2/A^2)^{1/2}$$

or

$$v = \omega(A^2 - x^2)^{1/2} \qquad ...(1.8)$$

and the acceleration of the particle is

$$a = \ddot{x} = -A\omega^2 \cos(\omega t - \phi) = -\omega^2 x \quad ...(1.9)$$

We see that, in simple harmonic motion, the acceleration is proportional to the displacement but opposite in sign.

If T is the kinetic energy, V the potential energy, then from the law of conservation of energy, in the absence of any friction-type losses, we have

$$E = T + V = \text{constant}$$

where E is the total energy of the oscillator.

Also, Force $\vec{F} = -\vec{\nabla} V$

or $-\dfrac{dV}{dx} = -kx$

or $V = \dfrac{1}{2}kx^2 + c$

or $$V = \frac{1}{2}m\omega^2 A^2 \cos^2(\omega t - \phi) + c \quad ...(1.10)$$

where c is an arbitrary constant.

The kinetic energy of the oscillator is

$$T = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t - \phi) \quad ...(1.11)$$

If $V = 0$ when $x = 0$, then $c = 0$ and

$$E = \frac{1}{2}m\omega^2 A^2 \quad ...(1.12)$$

(*i*) At the end points $x = \pm A$,
The velocity of the particle $v = 0$,
Acceleration $a = \omega^2 A$ directed towards the mean position,
kinetic energy $T = 0$
potential energy $V = \dfrac{1}{2}m\omega^2 A^2 = E$

(*ii*) At the mid-point ($x = 0$),
$v = \omega A$, $a = 0$, $T = \dfrac{1}{2}m\omega^2 A^2 = E$, $V = 0$

(*iii*) At $x = \pm A/\sqrt{2}$, $T = V = E/2$.

1.8 REFERENCE CIRCLE

Suppose that the point Q is moving anticlockwise with uniform angular velocity ω along a circular path with O as the centre (Fig. 1.2). This circle is called the reference circle for simple harmonic motion. BOB' is any diameter of the circle. $B'OB$ is chosen to be along the x-axis. From Q, a perpendicular QP is dropped on the diameter $B'B$. When Q moves with uniform angular velocity along the circular path, the point P executes simple harmonic motion along the diameter BB'. The amplitude of the back and forth motion of the point P

about the centre O is OB = the radius of the circle = A. Suppose Q is at B at time $t = 0$ and it takes a time t for going from B to Q and by this time the point P moves form B to P. If $\angle QOB = \theta$, $t = \theta/\omega$ or, $\theta = \omega t$, and $x = OP = OQ \cos\theta = A \cos \omega t$.

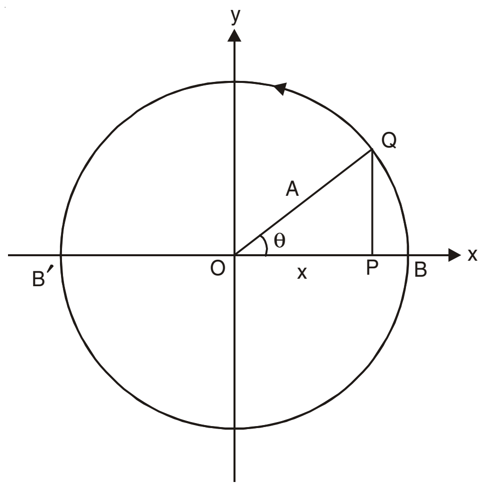

Fig. 1.2

When Q completes one revolution along the circular path, the point P executes one complete oscillation. The time period of oscillation $T = 2\pi/\omega$. If we choose the circle in the xy plane, the position of Q at any time t is given by

$$\vec{r} = A \cos \omega t\, \hat{i} + A \sin \omega t\, \hat{j}.$$

1.9 THE SIMPLE PENDULUM

The bob of the simple pendulum undergoes nearly SHM if its angle of swing is not large. The time period of oscillation of a simple pendulum of length l is given by

$$T = 2\pi\sqrt{l/g} \qquad \text{...(1.13)}$$

where g is the acceleration due to gravity.

1.10 ANGULAR SIMPLE HARMONIC MOTION (TORSIONAL PENDULUM)

A disc is suspended by a wire. If we twist the disc from its rest position and release it, it will oscillate about that position in angular simple harmonic motion. Twisting the disc through an angle θ in either direction, introduces a restoring torque

$$\Gamma = -C\theta \qquad \text{...(1.14)}$$

and the period of angular simple harmonic oscillator or torsional pendulum is given by

$$T = 2\pi\sqrt{I/C} \qquad \text{...(1.15)}$$

where I is the rotational inertia of the oscillating disc about the axis of rotation and C is the restoring torque per unit angle of twist.

SOLVED PROBLEMS

1. *A point is executing SHM with a period πs. When it is passing through the centre of its path, its velocity is 0.1 m/s. What is its velocity when it is at a distance of 0.03 m from the mean position?*

Solution

When the point is at a distance x from the mean position its velocity is given by Eqn. (1.8):

$$v = \omega(A^2 - x^2)^{1/2}.$$

Its time period, $T = 2\pi/\omega = \pi$; thus $\omega = 2\ s^{-1}$. *At* $x = 0$, $v = A\omega = 0.1$; thus $A = 0.05$ m. When $x = 0.03$ m, $v = 2\ [(0.05)^2 - (0.03)^2]^{1/2} = 0.08$ m/s.

2. *A point moves with simple harmonic motion whose period is 4 s. If it starts from rest at a distance 4.0 cm from the centre of its path, find the time that elapses before it has described 2 cm and the velocity it has then acquired. How long will the point take to reach the centre of its path?*

Solution

Amplitude A = 4 cm and time period $T = 2\pi/\omega = 4$ s. The distance from the centre of the path $x = 4–2 = 2$ cm. Since $x = A \cos \omega t$, we have $2 = 4 \cos \omega t$. Hence $t = 2/3$ s and the velocity $v = \omega \sqrt{A^2 - x^2} = \pi/2 \sqrt{4^2 - 2^2} = \pi\sqrt{3}$ cm/s. At the centre of the path $x = 0$ and $\omega t = \pi/2$ or, $t = 1$ s.

3. *A mass of 1 g vibrates through 1 mm on each side of the middle point of its path and makes 500 complete vibrations per second. Assuming its motion to be simple harmonic, show that the maximum force acting on the particle is π^2 N.*

Solution

$A = 1$ mm $= 10^{-3}$ m, $\nu = 500$ Hz and $\omega = 2\pi\nu$.

Maximum acceleration $= \omega^2 A$.

Maximum force $= m\omega^2 A = 10^{-3} \times 4\pi^2\ (500)^2 \times 10^{-3} = 2\ \pi^2 N$.

4. *At $t = 0$, the displacement of a point $x\ (0)$ in a linear oscillator is –8.6 cm, its velocity $v\ (0) = -\ 0.93$ m/s and its acceleration $a\ (0)$ is + 48 m/s^2. (a) What are the angular frequency ω and the frequency ν ? (b) What is the phase constant? (c) What is the amplitude of the motion?*

Solution

(*a*) The displacement of the particle is given by

$x(t) = A \cos(\omega t + \phi)$

Hence,
$$x(0) = A \cos \phi = -\,8.6 \text{ cm} = -\,0.086 \text{ m}$$
$$v(0) = -\omega A \sin \phi = -\,0.93 \text{ m/s}$$
$$a(0) = -\omega^2 A \cos \phi = 48 \text{ m/s}^2$$

Thus,
$$\omega = \sqrt{-\frac{a\,(0)}{x\,(0)}} = \sqrt{\frac{48}{0.086}} = 23.62 \text{ rad/s}$$

$$\nu = \omega/2\pi = \frac{23.62}{2\pi} = 3.76 \text{ Hz}$$

(*b*) $\dfrac{v(0)}{x(0)} = -\omega \tan \phi$

or
$$\tan \phi = -\frac{v(0)}{\omega x(0)} = -\frac{0.93}{23.62 \times 0.086} = -0.458$$

Hence $\phi = 155.4°, 335.4°$ in the range $0 \le \phi < 2\pi$. We shall see below how to choose between the two values.

(*c*) $A = \dfrac{x(0)}{\cos \phi} = \dfrac{-0.086}{\cos \phi}$.

The amplitude of the motion is a *positive* constant. So, $\phi = 335.4°$ cannot be the correct phase. We must therefore have

$$\phi = 155.4°$$

$$A = \frac{-0.086}{-0.909} = 0.0946 \text{ m.}$$

5. *A point performs harmonic oscillations along a straight line with a period T = 0.8 s and an amplitude A = 8 cm. Find the mean velocity of the point averaged over the time interval during which it travels a distance A/2, starting from (i) the extreme position, (ii) the equilibrium position.*

Solution

We have

$$x(t) = A \cos(\omega t - \phi)$$

(*i*) The particle moves from $x = A$ to $x = A/2$,

or
$$\omega t - \phi = 0 \text{ to } \omega t - \phi = \frac{\pi}{3},$$

or
$$t = \frac{\phi}{\omega} \text{ to } t = \frac{\phi}{\omega} + \frac{\pi}{3\omega}.$$

The average value of velocity over this interval is

$$< v > = \frac{1}{\pi / 3\omega} \int_{\phi/\omega}^{\phi/\omega + \pi/3\omega} \dot{x}\, dt$$

$$= \frac{3A\,\omega^2}{\pi} \left[\frac{\cos(\omega t - \phi)}{\omega} \right]_{t = \frac{\phi}{\omega}}^{t = \frac{\phi}{\omega} + \frac{\pi}{3\omega}}$$

$$= \frac{3A\,\omega}{\pi} \left(\frac{1}{2} - 1 \right) = -\frac{3A}{T}.$$

(*ii*) The particle moves from $x = 0$ to $x = A/2$

or,
$$t = \frac{\phi}{\omega} + \frac{\pi}{2\omega} \text{ to } t = \frac{\phi}{\omega} + \frac{\pi}{3\omega}$$

$$< v > = \frac{6A}{T}$$

The magnitude of the average velocity is

(*i*) $$\frac{3A}{T} = \frac{3\times 8}{0.8} \text{ cm/s} = 30 \text{ cm/s}$$

(*ii*) $$\frac{6A}{T} = 60 \text{ cm/s}$$

6. *A particle performs harmonic oscillations along the x-axis according to the law*

$$x = A \cos \omega t.$$

Assuming the probability P of the particle to fall within an interval from –A to A to be equal to unity, find how the probability density dP/dx depends on x. Here dP denotes the probability of the particle within the interval from x to x + dx.

Solution

The velocity of the particle at any time t is

$$\dot{x} = -A\omega \sin \omega t.$$

Time taken by the particle in traversing a distance from x to $x + dx$ is

$$\frac{dx}{|\dot{x}|} = \frac{dx}{A\omega\sqrt{1-x^2/A^2}} = \frac{dx}{\omega\sqrt{A^2-x^2}}.$$

Time taken by the particle in traversing the distance $-A$ to A is $T/2$.

Thus, $$dP = \frac{1}{T/2}\frac{dx}{\omega\sqrt{A^2-x^2}} = \frac{dx}{\pi\sqrt{A^2-x^2}}.$$

Hence $$\frac{dP}{dx} = \frac{1}{\pi\sqrt{A^2-x^2}}.$$

7. *In a certain engine a piston executes vertical SHM with amplitude 2 cm. A washer rests on the top of the piston. If the frequency of the piston is slowly increased, at what frequency will the washer no longer stay in contact with the piston?*

Solution

The maximum downward acceleration of the washer = g. If the piston accelerates downward greater than this, this washer will lose contact.

The largest downward acceleration of the piston

$$= \omega^2 A = \omega^2 \times 0.02 \text{ m/s}^2.$$

The washer will just separate from the piston

when $$\omega^2 \times 0.02 = g = 9.8 \text{ m/s}^2.$$

Thus, $$\nu = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{9.8}{0.02}} = 3.52 \text{ Hz}.$$

8. *A light spring of relaxed length a_0 is suspended from a point. It carries a mass m at its lower free end which stretches it through a distance l. Show that the vertical oscillations of the system are simple harmonic in nature and have time period, $T = 2\pi\sqrt{l/g}$.*

Solution

The spring is elongated through a distance l due to the weight mg. Thus we have

$$kl = mg$$

where k is the spring constant. Now the mass is further pulled through a small distance from its equilibrium position and released. When it is at a distance x from the mean position (Fig. 1.3), the net upward force on the mass m is

$$k(l + x) - mg = kx = mgx/l.$$

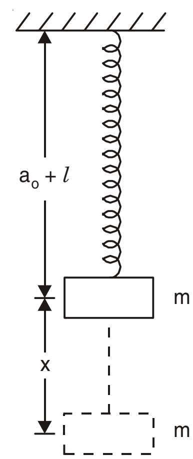

Fig. 1.3

Upward acceleration = $gx/l = \omega^2 x$, which is proportional to x and directed opposite to the direction of increasing x. Hence the motion is simple harmonic and its time period of oscillation is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{l/g}.$$

Note: Young's modulus of the material of the wire is given by

$$Y = \frac{mg}{A}/(l/L) = \frac{mgL}{Al},$$

where L is the length of the wire and A is the cross-sectional area of the wire.

Thus, $\dfrac{mg}{l} = \dfrac{AY}{L} = k$ = spring constant of the wire.

9. *A 100 g mass vibrates horizontally without friction at the end of an horizontal spring for which the spring constant is 10 N/m. The mass is displaced 0.5 cm from its equilibrium and released. Find: (a) Its maximum speed, (b) Its speed when it is 0.3 cm from equilibrium. (c) What is its acceleration in each of these cases?*

Solution

(*a*) $\omega = \sqrt{k/m} = \sqrt{10/0.1} = 10\ \text{s}^{-1}$ and $A = 0.005$ m.

The maximum speed = $A\omega = 0.05$ m/s

(*b*) $|v| = \omega\sqrt{A^2 - x^2} = 0.04$ m/s

(*c*) Acceleration $a = -\omega x$

(*i*) At $x = 0$, $a = 0$

(*ii*) At $x = 0.3$ cm, $a = -0.03\ \text{m/s}^2$.

10. *A mass M attached to a spring oscillates with a period of 2 s. If the mass is increased by 2 kg, the period increases by one second. Find the initial mass M assuming that Hooke's law is obeyed.* (I.I.T. 1979)

Solution

Since $T = 2\pi\sqrt{m/k}$, we have in the first case $2 = 2\pi\sqrt{M/k}$ and in the second case $3 = 2\pi\sqrt{(M+2)/k}$. Solving for M from these two equations we get $M = 1.6$ kg.

11. *Two masses m_1 and m_2 are suspended together by a massless spring of spring constant k as shown in Fig. 1.4. When the masses are in equilibrium, m_1 is removed without disturbing the system. Find the angular frequency and amplitude of oscillation.*(I.I.T. 1981)

Solution

When only the mass m_2 is suspended let the elongation of the spring be x_1. When both the masses $(m_2 + m_1)$ together are suspended, the elongation of the spring is $(x_1 + x_2)$.

Thus, we have

$$m_2 g = kx_1$$

$$(m_1 + m_2)g = k(x_1 + x_2)$$

where k is the spring constant.

Hence $$m_1 g = kx_2.$$

Thus, x_2 is the elongation of the spring due to the mass m_1 only. When the mass m_1 is removed the mass m_2 executes SHM with the amplitude x_2.

Amplitude of vibration = $$x_2 = m_1 g/k$$

Angular frequency $$\omega = \sqrt{k/m_2}.$$

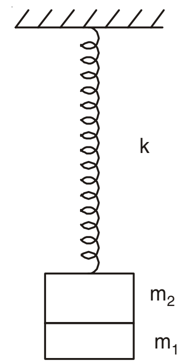

Fig. 1.4

12. *The 100 g mass shown in Fig. 1.5 is pushed to the left against a light spring of spring constant k = 500 N/m and compresses the spring 10 cm from its relaxed position. The system is then released and the mass shoots to the right. If the friction is ignored how fast will the mass be moving as it shoots away?*

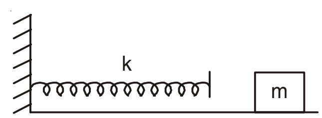

Fig. 1.5

Solution

When the spring is compressed the potential energy stored in the spring is

$$\frac{1}{2}kx^2 = \frac{1}{2} \times 500 \times (0.1)^2 = 2.5 \text{ J}.$$

After release this energy will be given to the mass as kinetic energy. Thus

$$\frac{1}{2} \times 0.1 \times v^2 = 2.5$$

from which $v = \sqrt{50} = 7.07$ m/s.

13. *In Fig. 1.6 the 1 kg mass is released when the spring is unstretched (the spring constant k = 400 N/m). Neglecting the inertia and friction of the pulley, find (a) the amplitude of the resulting oscillation, (b) its centre point of oscillation, and (c) the expressions for the potential energy and the kinetic energy of the system at a distance y downward from the centre point of oscillation.*

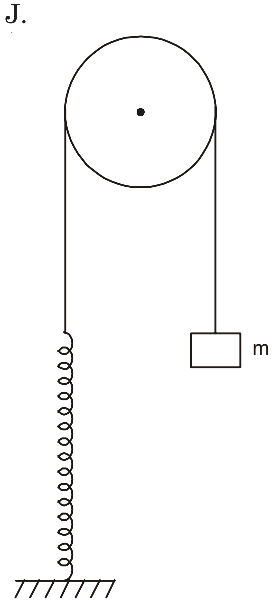

Fig. 1.6

Solution

(*a*) Suppose the mass falls a distance h before stopping. The spring is elongated by h. At this moment the gravitational potential energy (mgh) the mass lost is stored in the spring.

Thus, $$mgh = \frac{1}{2}kh^2$$

or $$h = \frac{2mg}{k} = \frac{2 \times 1 \times 9.8}{400} = 0.049 \text{ m}.$$

After falling a distance h the mass stops momentarily, its kinetic energy $T = 0$ at that moment and the PE of the system $V = 1/2\ kh^2$, and then it starts moving up. The mass will stop in its upward motion when the energy of the system is recovered as the gravitational PE (mgh). Therefore, it will rise 0.049 m above its lowest position. The amplitude of oscillation is thus 0.049/2 = 0.0245 m.

(b) The centre point of motion is at a distance $h/2$ = 0.0245 m below the point from where the mass was released.

(c) Total energy of the system

$$E = mgh = \frac{1}{2}kh^2.$$

At a distance y downward from the centre point of oscillation, the spring is elongated by ($h/2 + y$) and the total potential energy of the system is

$$V = \frac{1}{2}k\left(\frac{h}{2}+y\right)^2 + mg\left(\frac{h}{2}-y\right) = \frac{1}{2}k\left(y^2 + \frac{3}{4}h^2\right)$$

and the kinetic energy

$$T = E - V = \frac{1}{2}k\left(\frac{1}{4}h^2 - y^2\right), \quad -\frac{h}{2} \le y \le \frac{h}{2}.$$

14. *A linear harmonic oscillator of force constant 2×10^6 N/m and amplitude 0.01 m has a total mechanical energy of 160 J. Show that its (a) maximum potential energy is 160 J (b) maximum kinetic energy is 100 J.* (I.I.T. 1989)

Solution

From Eqns. (1.10) to (1.12), we have total mechanical energy = $1/2\ kA^2 + c$

$$= \frac{1}{2} \times 2 \times 10^6 \times (0.01)^2 + c = 100 \text{ J} + c = 160 \text{ J}$$

(a) Maximum P.E. = $\frac{1}{2}kA^2 + c$ = 160 J

(b) Maximum K.E. = $\frac{1}{2}kA^2$ = 100 J.

15. *A long light piece of spring steel is clamped at its lower end and a 1 kg ball is fastened to its top end (Fig. 1.7). A force of 5 N is required to displace the ball 10 cm to one side as shown in the figure. Assume that the system executes SHM when released. (a) Find the force constant of the spring for this type of motion. (b) Find the time period with which the ball vibrates back and forth.*

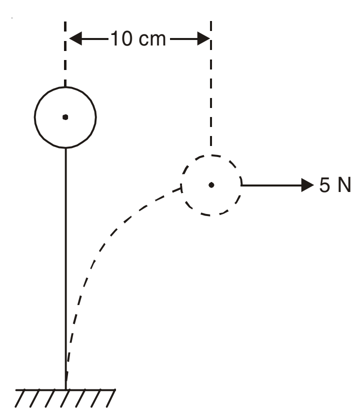

Fig. 1.7

Solution

(a) $k = \dfrac{\text{External Force}}{\text{Displacement}} = \dfrac{5\text{ N}}{0.1\text{ m}} = 50$ N/m

(b) $T = 2\pi\sqrt{m/k} = 2\pi\sqrt{1/50} = 0.89$ s.

16. *Two blocks (m = 1.0 kg and M = 11 kg) and a spring (k = 300 N/m) are arranged on a horizontal, frictionless surface as shown in Fig. 1.8. The coefficient of static friction between the two blocks is 0.40. What is the maximum possible amplitude of the simple harmonic motion if no slippage is to occur between the blocks?*

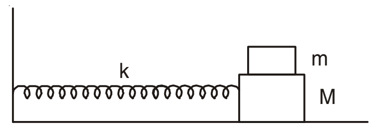

Fig. 1.8

Solution

Angular frequency of SHM = $\omega = \sqrt{300/12}$

Maximum force on the smaller body without any slippage is $m\omega^2 A = \mu mg$

Thus, $$A = \frac{\mu g}{\omega^2} = \frac{0.4 \times 9.8 \times 12}{300}\text{ m} = 15.68\text{ cm.}$$

17. *Two identical springs have spring constant k = 15 N/m. A 300 g mass is connected to them as shown in Figs. 1.9(a) and (b).*

Find the period of motion for each system. Ignore frictional forces.

Solution

(a) When the mass m is given a displacement x, one spring will be elongated by x, and the other will be compressed by x. They will each exert a force of magnitude kx on the mass in the direction opposite to the displacement. Hence, the total restoring force $F = -2\,kx = m\,\ddot{x}$. So,

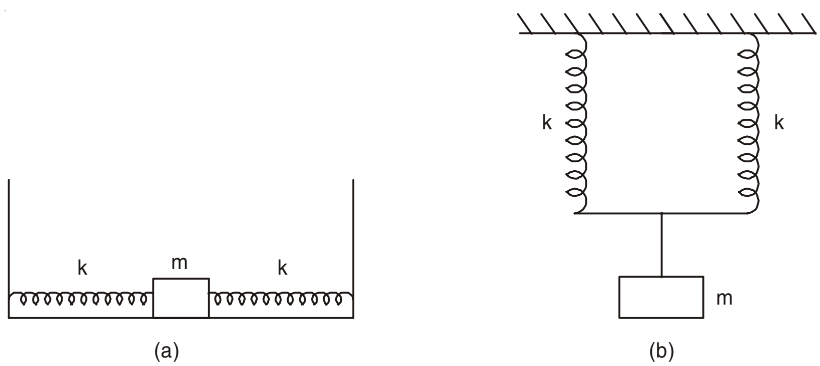

Fig. 1.9

$$\omega = \sqrt{2k/m} = \sqrt{2 \times 15/0.3} = 10\text{ s}^{-1}$$

$$T = 2\pi/\omega = 0.63\text{ s.}$$

(b) When the mass is pulled a distance y downward, each spring is stretched a distance y. The net restoring force on the mass = $-2\,ky$, $\omega = \sqrt{2k/m}$ and the period is also 0.63 s.

18. *Two massless springs A and B each of length a_0 have spring constants k_1 and k_2. Find the equivalent spring constant when they are connected in (a) series and (b) parallel as shown in Fig. 1.10 and a mass m is suspended from them.*

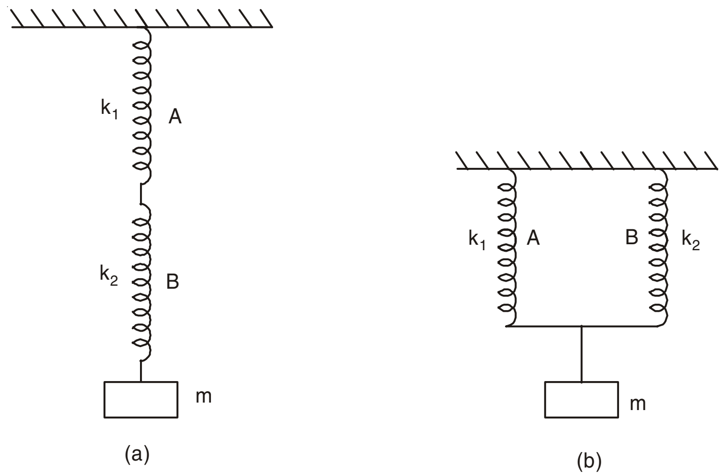

Fig. 1.10

Solution

(a) Let x_1 and x_2 be the elongations in springs A and B respectively. Total elongation $= x_1 + x_2$.

$$mg = k_1 x_1 \text{ and } mg = k_2 x_2$$

Thus,
$$x_1 + x_2 = mg\left(\frac{1}{k_1} + \frac{1}{k_2}\right).$$

If k is the equivalent spring constant of the combination (a), we have

$$x_1 + x_2 = mg/k$$

or
$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} \quad or, \; k = \frac{k_1 k_2}{k_1 + k_2}.$$

(b) Let x be the elongation in each spring.

$$mg = (k_1 + k_2)x$$

If k is the equivalent spring constant of the combination (b), we have

$$mg = kx$$

Thus,
$$k = k_1 + k_2.$$

19. *Two light springs of force constants k_1 and k_2 and a block of mass m are in one line AB on a smooth horizontal table such that one end of each spring is on rigid supports and the other end is free as shown in Fig. 1.11. The distance CD between the free ends of the springs is 60 cm. If the block moves along AB with a velocity 120 cm/s in between the springs, calculate the period of oscillation of the block.*

(k_1 = 1.8 N/m, k_2 = 3.2 N/m, m = 200 g) (I.I.T. 1985)

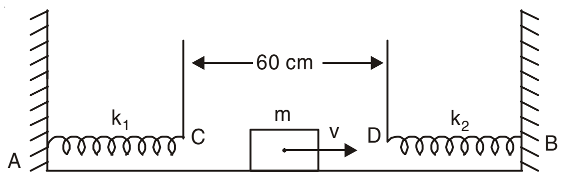

Fig. 1.11

Solution

The time period of oscillation of the block = time to travel 30 cm to the right from mid-point of CD + time in contact with the spring k_2 + time to travel DC (60 cm) to the left + time in contact with spring k_1 + time to travel 30 cm to the right from C

$$= \frac{30}{120} + \frac{1}{2}\left[2\pi\sqrt{\frac{m}{k_2}}\right] + \frac{60}{120} + \frac{1}{2}\left[2\pi\sqrt{\frac{m}{k_1}}\right] + \frac{30}{120}$$

$$= 1 + \pi\left[\sqrt{0.2/3.2} + \sqrt{0.2/1.8}\right] = 1 + \pi\left[\frac{1}{4} + \frac{1}{3}\right]$$

$$= 2.83 \text{ s.}$$

20. *The mass m is connected to two identical springs that are fixed to two rigid supports (Fig. 1.12). Each of the springs has zero mass, spring constant k, and relaxed length a_0. They each have length a at the equilibrium position of the mass. The mass can move in the x-direction (along the axis of the springs) to give longitudinal oscillations. Find the period of motion. Ignore frictional forces.*

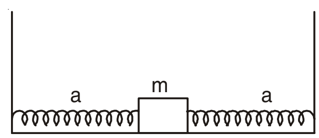

Fig. 1.12

Solution

At the equilibrium position each spring has tension $T_0 = k(a - a_0)$. Let at any instant of time x be the displacement of the mass from the equilibrium position. At that time the net force on the mass due to two springs in the +*ve* x-direction is

$$F_x = -k(a + x - a_0) + k(a - x - a_0) = -2kx.$$

Thus, $$m\ddot{x} = -2kx \text{ and } \omega^2 = 2k/m$$

and $$T = 2\pi\sqrt{m/(2k)}.$$

21. *A mass m is suspended between rigid supports by means of two identical springs. The springs each have zero mass, spring constant k, and relaxed length a_0. They each have length a at the equilibrium position of mass m [Fig. 1.13(a)]. Consider the motion of the mass along the y-direction (perpendicular to the axis of the springs) only. Find the frequency of*

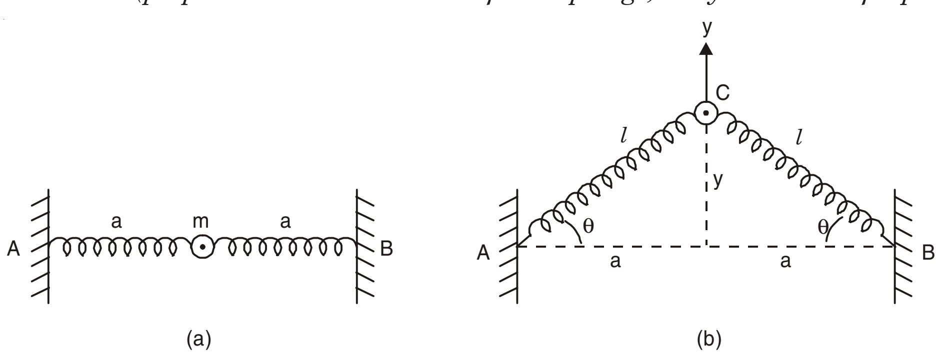

Fig. 1.13

transverse oscillations of the mass under (a) slinky approximation ($a_0 << a$), (b) small oscillations approximation ($y << a$).

Solution

At equilibrium each spring exerts tension $T_0 = k\,(a - a_0)$. In the general configuration (Fig. 1.13(*b*)) each spring has length l and tension $T = (l - a_0)$ which is exerted along CA or CB. The y-component of this force is $-T \sin\theta$. Each spring contributes a return force $T \sin\theta$ in the –ve y–direction. Using Newton's second law, we have

$$m\ddot{y} = -2T\sin\theta = -2k(l - a_0)y/l. \qquad \ldots(1.16)$$

The x-components of the two forces due to two springs balance each other so that there is no motion along the x-direction. Thus, we have

$$m\ddot{y} = -2ky\left(1 - \frac{a_0}{\sqrt{a^2 + y^2}}\right). \qquad \ldots(1.17)$$

The above equation is not exactly in the form that gives rise to SHM.

(*a*) Slinky approximation ($a_0/a << 1$): Since $l > a$, $a_0/l << 1$ and we get from Eqn. (1.16)

$$\ddot{y} = -\frac{2k}{m}y = -\omega^2 y$$

The time period is same as longitudinal oscillation. Time period = $2\pi\sqrt{m/2k}$.

(*b*) Small oscillations approximation ($y << a$): Under this approximation, we have

$$\frac{a_0}{\sqrt{a^2 + y^2}} \approx \frac{a_0}{a}\left(1 - \frac{y^2}{2a^2}\right).$$

Thus,
$$m\ddot{y} = -2ky\left(1 - \frac{a_0}{a} + \frac{a_0 y^2}{2a^3}\right).$$

we neglect $(y/a)^3$ term in this equation, we get

$$\ddot{y} = -\frac{2ky}{ma}(a - a_0) = -\frac{2T_0}{ma}y.$$

Hence
$$\omega^2_{tr} = \frac{2T_0}{ma} = \frac{2k}{ma}(a - a_0) = \frac{2k}{m}\left(1 - \frac{a_0}{a}\right)$$

and
$$\text{time period} = \frac{2\pi\sqrt{m/2k}}{\sqrt{1 - a_0/a}} = 2\pi\sqrt{ma/2T_0}\ .$$

22. *A ball of mass m is connected to rigid walls by means of two wires of lengths l_1 and l_2 (Fig. 1.14). At equilibrium the tension in each wire is T_0. The mass m is displaced slightly from equilibrium in the vertical direction and released. Determine the frequency for small oscillations.*

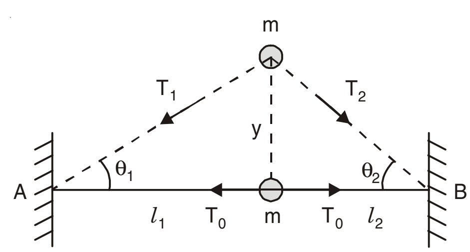

Fig. 1.14

Solution

Restoring force = $T_1 \sin\theta_1 + T_2 \sin\theta_2$.

For small displacements, $T_1 \approx T_0$ and $T_2 \approx T_0$,

$\sin\theta_1 \approx \tan\theta_1 = y/l_1$, $\sin\theta_2 \approx \tan\theta_2 = y/l_2$.

Thus, $$m\ddot{y} = -T_0\left(\frac{y}{l_1}+\frac{y}{l_2}\right) = -T_0\frac{l_1+l_2}{l_1 l_2}y$$

and $$\omega = \left[\frac{T_0(l_1+l_2)}{ml_1 l_2}\right]^{1/2}.$$

23. *A vertical spring of length 2L and spring constant k is suspended at one end. A body of mass m is attached to the other end of the spring. The spring is compressed to half its length and then released. Determine the kinetic energy of the body, and its maximum value, in the ensuing motion in the presence of the gravitational field.*

Solution

If the position of the body is measured from the relaxed position of the spring by the coordinate y (positive upward) and if the P.E. V is set equal to zero at $y = 0$, we have

$$V = \frac{1}{2}ky^2 + mgy,$$

$$T + V = T + \frac{1}{2}ky^2 + mgy = E = \text{Total energy},$$

where T is the K.E. of the body.

Now, $T = 0$ when $y = L$, or, $E = \frac{1}{2}kL^2 + mgL.$

Thus, $T = \frac{1}{2}k\left(L^2 - y^2\right) + mg\,(L - y).$

T is maximum when $\frac{dT}{dy} = 0$ or, $y = -\frac{mg}{k}$,

and $T_{max} = \frac{1}{2}k(L+mg/k)^2.$

24. *Find the time period of a simple pendulum.*

Solution

A small bob of mass m is attached to one end of a string of negligible mass and the other end of the string is rigidly fixed at O (Fig. 1.15). OA is the vertical position of the simple pendulum of length l and this is also the equilibrium position of the system. The pendulum can oscillate only in the vertical plane and at any instant of time B is the position of the bob. Let $\angle AOB = \psi$. The displacement of the bob as measured along the perimeter of the circular arc of its path is $AB = l\psi$. The instantaneous tangential velocity is $l\frac{d\psi}{dt}$ and the corresponding tangential acceleration is $l\frac{d^2\psi}{dt^2}$.

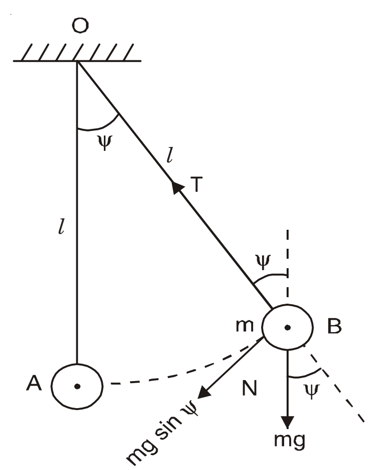

Fig. 1.15

The return force acting on the bob along the tangent BN drawn at B to the circular arc AB is $mg\sin\psi$. There is no component of the tension T of the string along BN. The return force $mg\sin\psi$ acts in a direction opposite to the direction of increasing ψ. Thus we have

$$ml\frac{d^2\psi}{dt^2} = -mg\sin\psi. \qquad \text{...(1.18)}$$

Maclaurin's series for $\sin\psi$ is

$$\sin\psi = \psi - \frac{\psi^3}{3!} + \frac{\psi^5}{5!} - \ldots$$

For sufficiently small ψ, $\sin\psi \approx \psi$ (in radians) and we have

$$\frac{d^2\psi}{dt^2} = -\omega^2\psi,$$

with

$$\omega^2 = g/l.$$

The motion is simple harmonic and its time period of oscillation is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{l/g}.$$

25. *What is the period of small oscillation of an ideal pendulum of length l, if it oscillates in a truck moving in a horizontal direction with acceleration a?*

Solution

Let the equilibrium position be given by the angle ϕ (Fig. 1.16). In this position the force on the mass m along the horizontal axis is equal to ma. The angle ϕ is determined by the equations.

$$T \sin \phi = ma, \; T \cos \phi = mg.$$

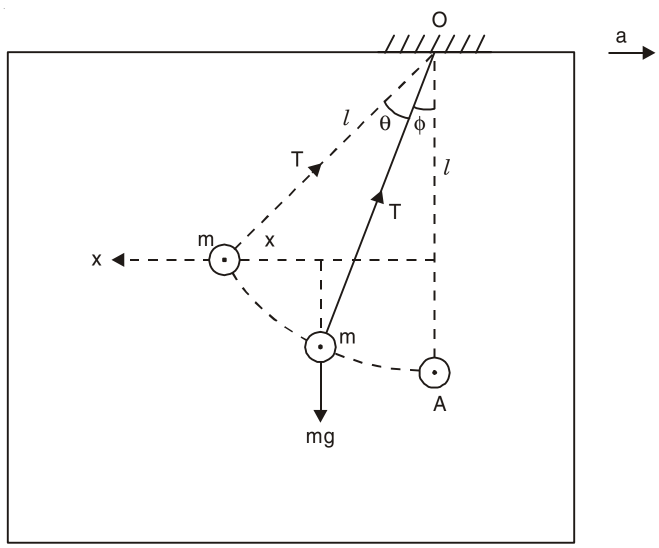

Fig. 1.16

When the pendulum is displaced by a small amount θ, it will perform simple harmonic motion around the equilibrium position. Its equation of motion is

$$m\ddot{x} = -T \sin(\theta + \phi)$$

where x is the distance from the vertical OA.

For small θ, $\sin(\theta + \phi) \approx \theta \cos \phi + \sin \phi$

$$= \theta\frac{mg}{T} + \frac{ma}{T}.$$

Thus

$$m\ddot{x} = -ma - mg\theta.$$

θ and l are related geometrically as

$x = l\sin(\theta + \phi) \approx l\,\theta\cos\phi + l\sin\phi$

or
$$\ddot{x} = l\cos\phi\,\ddot{\theta}$$

Hence
$$l\cos\phi\,\ddot{\theta} = -a - g\theta$$

or
$$\ddot{\theta} = -\frac{g}{l\cos\phi}\left(\theta + \frac{a}{g}\right).$$

If we make the following substitution

$$\psi = \theta + \frac{a}{g}$$

we get

$$\ddot{\psi} = -\frac{g}{l\cos\phi}\psi = -\omega^2\psi$$

with time period of oscillation

$$T = \frac{2\pi}{\omega} = 2\pi\left[\frac{l\cos\phi}{g}\right]^{1/2}.$$

Now,
$$\cos\phi = \frac{mg}{T} = \frac{mg}{\sqrt{m^2a^2 + m^2g^2}} = \frac{g}{\sqrt{a^2 + g^2}}$$

Thus,
$$T = 2\pi\left[\frac{l}{\sqrt{a^2+g^2}}\right]^{1/2}.$$

26. *A simple pendulum of bob mass m is suspended vertically from O by a massless rigid rod of length L (Fig. 1.17 (a)). The rod is connected to a spring of spring constant k at a distance h form O. The spring has its relaxed length when the pendulum is vertical.*

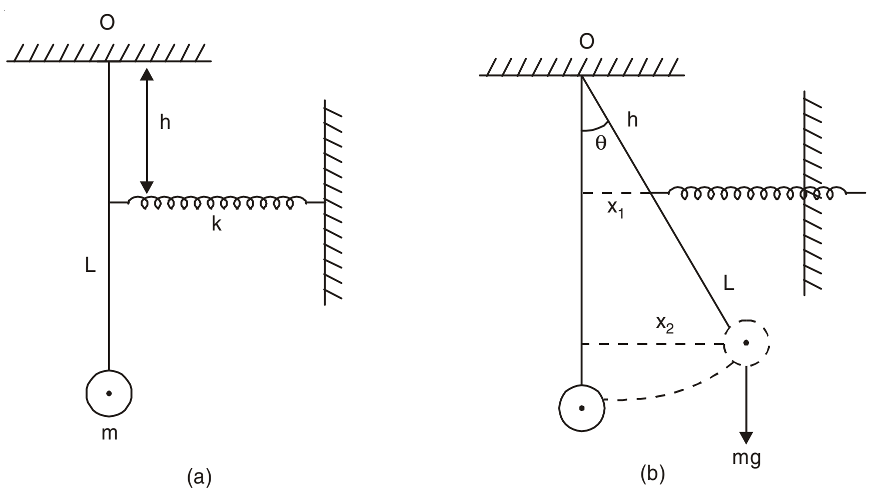

Fig. 1.17

Write the differential equation of motion and determine the frequency for small oscillations of this pendulum.

Solution

Let θ be a small deflection of the pendulum from its equilibrium position. The spring is compressed by x_1 and it exerts a force $F_s = kx_1$ on the rod. We have

$$x_1 = h \sin\theta \text{ and } x_2 = L \sin\theta.$$

Taking the sum of torques about the point O we obtain (for small deflection θ):

$$-F_s h - mgx_2 = mL^2\ddot{\theta}$$

or
$$mL^2\ddot{\theta} + (kh^2 + mgL)\sin\theta = 0.$$

Since $\sin\theta \approx \theta$ for small oscillations we get SHM with frequency

$$\omega = \left[\frac{kh^2 + mgL}{mL^2}\right]^{1/2} = \left[\frac{g}{L} + \frac{kh^2}{mL^2}\right]^{1/2}.$$

27. *A simple pendulum of bob mass m is suspended vertically from O by a massless rigid rod of length L. The rod is connected to two identical massless springs on two sides of the rod at a distance a from O (Fig. 1.18). The spring constant of each spring is k. The springs have their relaxed lengths when the pendulum is vertical.*

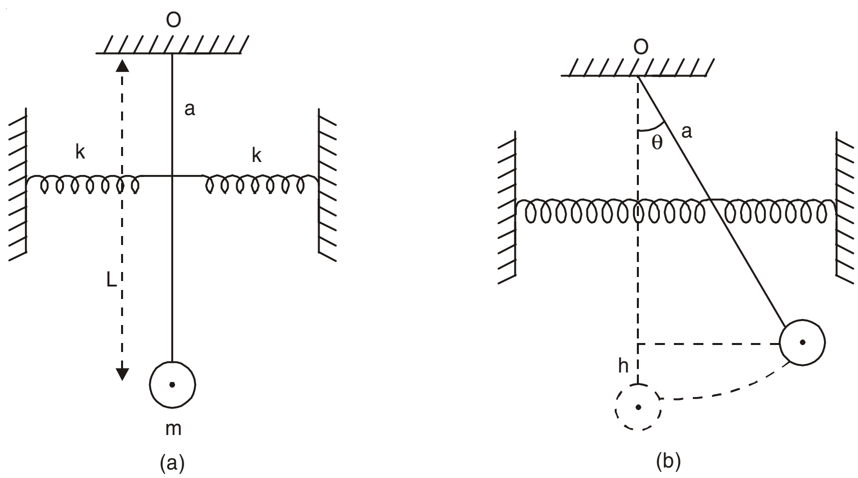

Fig. 1.18

Write the differential equation of motion and determine the frequency of small oscillations of this pendulum.

Solution

When the pendulum is at an angle θ with the vertical [Fig. 1.18(b)], the pendulum is raised by the distance $h = L - L\cos\theta$ and the PE of the pendulum is

$$(\text{P.E.})_m = mgh = mgL(1 - \cos\theta).$$

The zero level of the PE is chosen with the pendulum being vertical.

When the pendulum is at an angle θ, one of the springs is stretched by the amount $a\theta$, while the other is compressed by the same amount. The PE of the springs is

$$(\text{P.E.})_s = \frac{1}{2}k(a\theta)^2 \times 2 = ka^2\theta^2.$$

Thus the total P.E. of the system is

$$V = mgL(1 - \cos\theta) + ka^2\theta^2.$$

The kinetic energy is associated only with the mass m. The velocity $v = L\dot{\theta}$ and KE is

$$T = \frac{1}{2}mL^2\dot{\theta}^2$$

Thus the total energy of the system is

$$E = \tfrac{1}{2}mL^2\dot{\theta}^2 + mg\,L(1 - \cos\theta) + Ka^2\theta^2.$$

Since the total energy of the system is conserved, we have

$$\frac{dE}{dt} = mL^2\dot{\theta}\ddot{\theta} + mgL\,\sin\theta\,\dot{\theta} + 2ka^2\dot{\theta}\,\theta = 0$$

or
$$\ddot{\theta} + \frac{g}{L}\sin\theta + \frac{2ka^2}{mL^2}\theta = 0.$$

Since $\sin\theta \approx \theta$ for small oscillations we get SHM with frequency

$$\omega = \left[\frac{g}{L} + \frac{2ka^2}{mL^2}\right]^{\frac{1}{2}}$$

28. *A simple pendulum is suspended from a peg on a vertical wall. The pendulum is pulled away from the wall to a horizontal position (Fig. 1.19) and released. The ball hits the wall, the coefficient of restitution being $2/\sqrt{5}$. What is the minimum number of collisions after which the amplitude of oscillation becomes less than 60°?* (I.I.T. 1987)

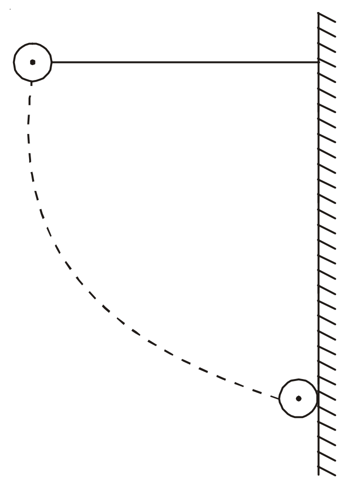

Fig. 1.19

Solution

Let v_0 be the velocity of the bob just before the first collision.

Then
$$\frac{1}{2}mv_0^2 = mgL$$

or
$$v_0 = \sqrt{2gL}\,.$$

The velocity of the bob just after the 1st collision is

$$v_1 = \frac{2}{\sqrt{5}}v_0 = \frac{2}{\sqrt{5}}\sqrt{2gL}\,.$$

v_1 will be the velocity of the bob just before 2nd collision. The velocity of the bob just after the second collision is

$$v_2 = \frac{2}{\sqrt{5}}v_1 = \left(\frac{2}{\sqrt{5}}\right)^2\sqrt{2gL}$$

The velocity just after the nth collision is

$$v_n = \left(\frac{2}{\sqrt{5}}\right)^n\sqrt{2gL}\,.$$

We assume that after n collisions the amplitude of oscillation becomes 60°.

Thus,

$$\frac{1}{2}mv_n^2 = mg\,(L - L\cos 60°) = \frac{1}{2}mgL$$

or
$$v_n^2 = gL$$

or
$$\left(\frac{2}{\sqrt{5}}\right)^{2n} 2gL = gL$$

or
$$\left(\frac{4}{5}\right)^n = \frac{1}{2}$$

n is slightly greater than 3. In fact

$$n = \frac{\log_{10} 2}{\log_{10} 5 - \log_{10} 4} = 3.1$$

Thus the minimum number of collisions after which the amplitude becomes less than 60° is 4.

29. *A bullet of mass M is fired with a velocity 50 m/s at an angle θ with the horizontal. At the highest point of its trajectory, it collides head-on with a bob of mass 3 M suspended by a massless string of length 10/3 m and gets embedded in the bob. After the collision the string moves through an angle of 120°. Find*

(*i*) *the angle θ*

(*ii*) *the vertical and horizontal coordinates of the initial position of the bob with respect to the point of firing of the bullet. (Take g = 10 m/s²)* (I.I.T 1988)

Solution

(*i*) At the highest point of the bullet the horizontal component of velocity = $u\cos\theta$ and the vertical component of velocity = 0. Let (x, y) be the coordinates of the initial position A of the bob (Fig. 1.20).

We have

$$y = \frac{u^2 \sin^2\theta}{2g}$$

$$y = \frac{u^2 \sin 2\theta}{2g}.$$

Due to head-on collision of the bullet with the bob at A we have from the conservation of linear momentum

$$Mu\cos\theta = 4Mv$$

where v is the initial velocity of the bob along the x-direction.

Thus, $$v = \frac{u}{4}\cos\theta.$$

At the highest point (B) of the path of the combined mass let the velocity be v_1. At this position, we have

$$4Mg\cos 60° = \frac{4Mv_1^2}{l}$$

or $$v_1^2 = \frac{gl}{2}$$

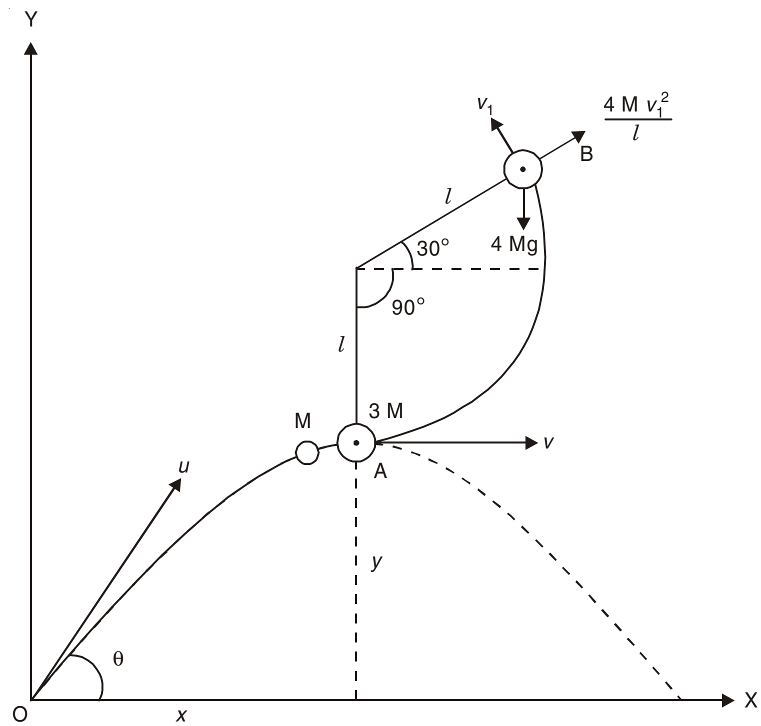

Fig. 1.20

From conservation of mechanical energy of the bob, we get

$$\frac{1}{2}(4M)v^2 = \frac{1}{2}(4M)\,v_1^2 + 4Mg(l + l\sin 30°)$$

or $$v^2 = v_1^2 + 3gl$$

Thus, $$\frac{u^2}{16}\cos^2\theta = \frac{gl}{2} + 3gl = \frac{7gl}{2}$$

or $$\cos^2\theta = \frac{56gl}{u^2} = \frac{56\times 10}{50\times 50}\times\frac{10}{3} = \frac{56}{75}$$

or $$\theta = 30.2°$$

(*ii*) $$y = \frac{u^2\sin^2\theta}{2g} = \frac{50\times 50\times(0.503)^2}{2\times 10} = 31.6\text{ m},$$

$$x = \frac{u^2\sin 2\theta}{2g} = \frac{50\times 50\times(0.869)}{2\times 10} = 108.7\text{ m}.$$

30. *Two identical balls A and B each of mass 0.1 kg are attached to two identical massless springs. The spring-mass system is constrained to move inside a rigid smooth pipe bent in the form of a circle as shown in Fig. 1.21. The pipe is fixed in a horizontal plane. The centres of the balls can move in a circle of radius 0.06 metre. Each spring has a natural length* *of 0.06 π metre and spring constant 0.1 N/m. Initially, both the balls are displaced by an angle $\theta = \pi/6$ radian with respect to the diameter PQ of the circle (as shown in the figure) and released from rest.*

(*i*) *Calculate the frequency of oscillation of ball B.*

(*ii*) *Find the speed of ball A when A and B are at two ends of the diameter PQ.*

(*iii*) *What is the total energy of the system?* (I.I.T. 1993)

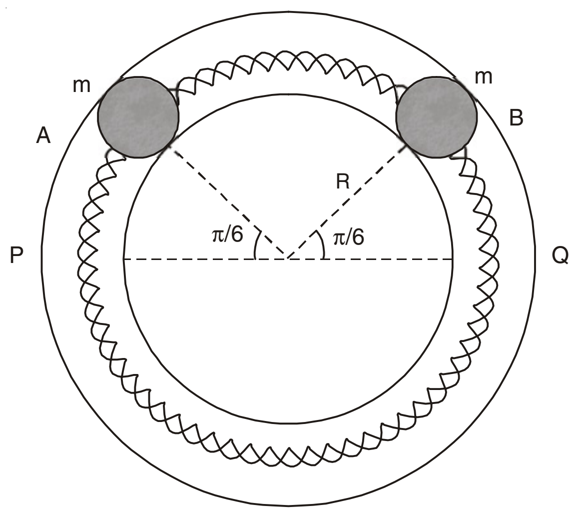

Fig. 1.21

Solution

(*i*) At an angular displacement θ of the balls the compression or extension in respective springs = 2 $R\theta$.

Thus, force on B due to both springs = 4 $kR\theta$, where k is the spring constant.

Now, $\dfrac{d^2\theta}{dt^2}$ = angular acceleration and $R\dfrac{d^2\theta}{dt^2}$ = linear acceleration.

The equation of motion of the mass B is given by

$$mR\frac{d^2\theta}{dt^2} = -4kR\theta$$

or
$$\frac{d^2\theta}{dt^2} = -\frac{4k}{m}\theta = -\omega^2\theta, \text{ which represents SHM.}$$

Thus,
$$\nu = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{4k}{m}} = \frac{1}{2\pi}\sqrt{\frac{4\times 0.1}{0.1}} = \frac{1}{\pi}\text{Hz}.$$

(*ii*)
$$[\text{K.E.}]_{\theta = 0} = [\text{P.E.}]_{\theta = \frac{\pi}{6}}$$

or
$$2 \times \frac{1}{2}mv^2 = 2 \times \frac{1}{2}k(2R\theta)^2$$

or
$$\nu = 2R\theta\sqrt{\frac{k}{m}} = 2 \times 0.06 \times \frac{\pi}{6} = 0.02\pi \text{ ms}^{-1}$$

(*iii*) Total energy $= [\text{P.E.}]_{\theta = \frac{\pi}{6}}$

$$= 4kR^2\theta^2$$

$$= 4 \times 0.1 \times 36 \times 10^{-4} \times \frac{\pi^2}{36}$$

$$= 4\pi^2 \times 10^{-5} \text{ J.}$$

31. *If the earth were a homogeneous sphere of radius R and a straight hole were bored in it through its centre, show that a body dropped into the hole will execute SHM. Find its time period.*

Solution

Suppose AB is a straight hole (Fig. 1.22) passing through the centre O of the earth. A body of mass m is dropped into the hole. At any instant of time the body is at C at a distance x from the centre of the earth. When the body is at C, the force of attraction on the body due to earth is

$$F = -G\frac{4}{3}\pi x^3 \frac{\rho m}{x^2}$$

where ρ, density of the material of earth, is assumed to be uniform everywhere and G, the universal gravitational constant.

At C, the acceleration of the body towards the centre O is

$$a = \frac{F}{m} = -G\frac{4}{3}\pi\rho x.$$

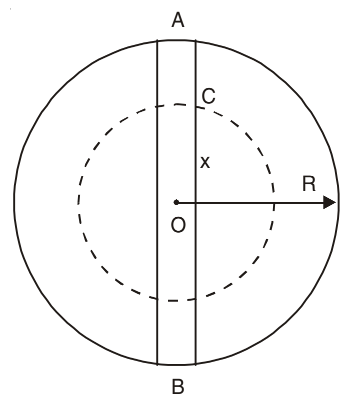

Fig. 1.22

As $a \propto x$ and it acts opposite to the direction of increasing x, the motion of the body is simple harmonic. We have

$$\omega^2 = G\frac{4}{3}\pi\rho.$$

Now the acceleration of the body on the surface of the earth is

$$g = G\frac{M}{R^2} = G\frac{4}{3}\pi\rho R$$

where M = mass of the earth.

Hence, $\omega^2 = \dfrac{g}{R}$

and the time period $T = 2\pi\sqrt{\dfrac{R}{g}}$.

with $R = 6.4 \times 10^6$ m, $g = 9.8$ m/s^2,

We have $T = 5077.6$ s.

32. *A cylindrical piston of mass M slides smoothly inside a long cylinder closed at one end, enclosing a certain mass of gas. The cylinder is kept with its axis horizontal. If the piston is disturbed from its equilibrium position, it oscillates simple harmonically. Show that the period of oscillation will be (Fig. 1.23)*

$$T = 2\pi\sqrt{\frac{Mh}{PA}} = \frac{2\pi}{A}\sqrt{\frac{MV}{P}} \qquad \text{(I.I.T. 1981)}$$

Solution

Suppose that the initial pressure of the gas is P and initial volume is $V = Ah$. The piston is moved isothermally from C to D through a distance x (Fig. 1.23). The gas inside the cylinder will be compressed and it will try to push the piston to its original position. When the piston is at D let the pressure of the gas be $P + \delta P$ and volume $= V - \delta V = V - Ax$. Since the process is isothermal, we have

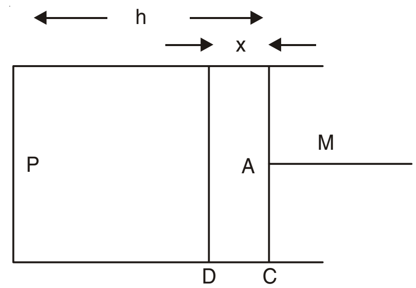

Fig. 1.23

$$PV = (P + \delta P)(V - \delta V) \approx PV - P\delta V + V\delta P$$

or
$$\delta P = \frac{P\delta V}{V} = \frac{PAx}{V}.$$

The return force acting on the piston is

$$A\delta P = \frac{A^2Px}{V}.$$

The acceleration a of the piston is proportional to x and directed opposite to the direction of increasing x:

$$a = -\frac{A^2Px}{MV}$$

Thus, the motion of the piston is simple harmonic and its time period is

$$T = 2\pi\sqrt{\frac{MV}{A^2P}} = \frac{2\pi}{A}\sqrt{\frac{MV}{P}} = 2\pi\sqrt{\frac{Mh}{PA}}.$$

33. *An ideal gas is enclosed in a vertical cylindrical container and supports a freely moving piston of mass M. The piston and the cylinder have equal cross-sectional area A. Atmospheric pressure is P_0 and when the piston is in equilibrium, the volume of the gas is V_0. The piston is now displaced slightly from the equilibrium position. Assuming that the system is completely isolated from its surroundings, show that the piston executes simple harmonic motion and find the frequency of oscillation.* (I.I.T. 1981)

Solution

Since the system is completely isolated from the surroundings, there will be adiabatic change in the container. Let the initial pressure and the volume of the gas be P and V_0 respectively. When the piston is moved down a distance x, the pressure increases to $P + \delta P$ and volume decreases to $V_0 - \delta V$. Thus,

$$\begin{aligned} PV_0^\gamma &= (P + \delta P)(V_0 - \delta V)^\gamma \\ &= (P + \delta P)V_0^\gamma\left(1 - \frac{\delta V}{V_0}\right)^\gamma \\ &\approx (P + \delta P)V_0^\gamma\left(1 - \gamma\frac{\delta V}{V_0}\right) \\ &\approx V_0^\gamma\left(P + \delta P - \frac{\gamma P\delta V}{V_0}\right) \end{aligned}$$

where γ is the ratio of specific heats at constant pressure and volume ($\gamma = C_p/C_v$).

Hence,
$$\delta P = \frac{\gamma P \delta V}{V_0} = \frac{\gamma P A x}{V_0}.$$

The acceleration of the piston is given by
$$a = -\frac{\gamma P A x}{V_0} \times \frac{A}{M} = -\frac{\gamma P A^2 x}{M V_0},$$

which shows that the piston executes SHM, with
$$\omega^2 = \frac{\gamma P A^2}{M V_0}$$

Now,
$$P = P_0 + \frac{Mg}{A} = \frac{AP_0 + Mg}{A}$$

The frequency of oscillation is given by
$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\left[\frac{\gamma A(AP_0 + Mg)}{M V_0}\right]^{1/2}.$$

34. *Two non-viscous, incompressible and immiscible liquids of densities ρ and 1.5 ρ are poured into two limbs of a circular tube of radius R and small cross-section kept fixed in a vertical plane as shown in Fig. 1.24. Each liquid occupies one-fourth the circumference of the tube. (a) Find the angle θ that the radius vector to the interface makes with the vertical in equilibrium position. (b) If the whole liquid is given a small displacement from its equilibrium position, show that the resulting oscillations are simple harmonic. Find the time period of these oscillations.* (I.I.T. 1991)

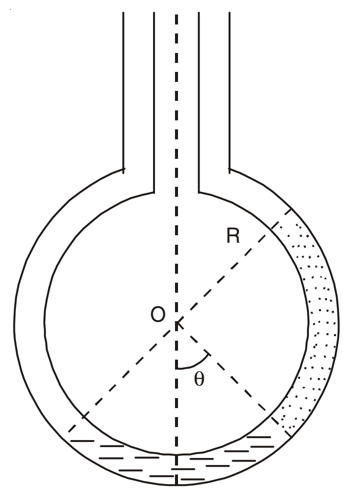

Fig. 1.24

Solution

(a) Since each liquid occupies one-fourth the circumference of the tube, $\angle AOC = 90° = \angle BOC$ [Fig. 1.25 (a)].

The pressure P_1 at D due to liquid on the left limb is
$$P_1 = (R - R \sin\theta)\, 1.5\, \rho g$$

The pressure P_2 at D due to liquid on the right limb is
$$P_2 = (R - R \cos\theta)\, 1.5\, \rho g + (R \sin\theta + R \cos\theta)\rho g$$

At equilibrium $P_1 = P_2$. Thus, we have
$$(1 - \sin\theta)\, 1.5 = (1 - \cos\theta)\, 1.5 + \sin\theta + \cos\theta$$

Solving this equation, we get $2.5 \sin\theta = 0.5 \cos\theta$,

or
$$\tan\theta = \frac{0.5}{2.5} = 0.2$$

or
$$\theta = 11.3°.$$

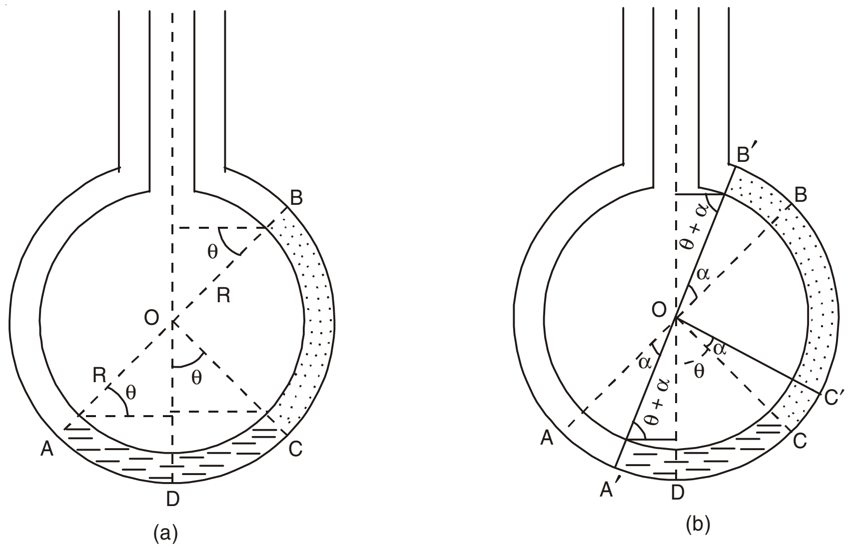

Fig. 1.25

(*b*) When the liquid is given a small upward displacement $y = BB'$ in the right limb [Fig. 1.25 (*b*)], then $y = R\alpha$ where $\alpha = \angle B'OB$, and A goes to A' and C goes to C'. The pressure difference at D is

$$\begin{aligned} dP &= P_2' - P_1' \\ &= [R - R\cos(\theta + \alpha)\ 1.5\ \rho g + [R\sin(\theta + \alpha) \\ &\qquad + [R\cos(\theta + \alpha)]\rho g - [R - R\sin(\theta + \alpha)]\ 1.\ 5\ \rho g \\ &= \mathrm{R}\rho g\ \{2.5\sin(\theta + \alpha) - 0.5\cos(\theta + \alpha)\} \\ &\approx \mathrm{R}\rho g\ \{2.5\ [\sin\theta + \alpha\cos\theta] - 0.5[\cos\theta - \alpha\sin\theta]\} \\ &= \mathrm{R}\rho g\ \{2.5\alpha\cos\theta + 0.5\alpha\sin\theta\} \\ &= y\rho g\ \{2.5\cos\theta + 0.5\sin\theta\} \\ &= 2.55\rho g y \end{aligned}$$

Thus, Restoring force $= -\ 2.55\ \rho g y \times A$,

where A is the area of cross-section of the tube.

Mass of the liquid in the tube is

$$m = \frac{2\pi R}{4} A\rho + \frac{2\pi R}{4} A \times 1.5\rho = 1.25\ \pi R A\rho.$$

The acceleration of the liquid column is

$$a = -\frac{2.55\rho g\ yA}{1.25\pi R A\rho} = -\,2.04\left(\frac{g}{\pi R}\right)y$$

which shows that the motion is simple harmonic.

The time period of oscillations is given by

$$T = 2\pi\sqrt{\frac{\pi R}{2.04g}} = 2.49\sqrt{R}\ \mathrm{s}\cdot$$

35. *Ten kg of mercury are poured into a glass U tube [Fig. 1.26]. The tube's inner diameter is 1.0 cm and the mercury oscillates freely up and down about its equilibrium position (x = 0). Calculate (a) the effective spring constant of motion, and (b) the time period. Ignore frictional and surface tension effects.*

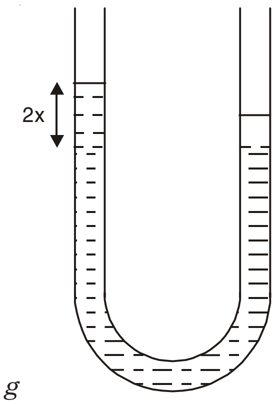

Fig. 1.26

Solution

(a) When the mercury is displaced x metres from its equilibrium position in the right arm, the restoring force is due to the weight of the unbalanced column of mercury of weight $2x$. Now,

$$\text{Weight} = \text{Volume} \times \text{Density} \times g = (\pi r^2 2x) \times \rho \times g$$

where $$\rho = 13.6 \text{ g/cm}^3 = \frac{13.6 \times 10^{-3}\,\text{kg}}{10^{-6}\,\text{m}^3} = 13.6 \times 10^3 \text{ kg/m}^3.$$

Thus, the restoring force = $-(2\pi r^2 \rho g)\, x$, and Hooke's law is valid, thereby we see that the effective spring constant for the system is

$$k = 2\pi r^2 \rho g = 2\pi(0.005)^2 (13.6 \times 10^3) \times 9.8 = 20.94 \text{ N/m}$$

(b) $$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{10}{20.94}} = 4.34 \text{ s.}$$

36. *Two identical positive point charges + Q each, are fixed at a distance of 2a apart. A point charge + q lies midways between the fixed charges. Show that for a small displacement along the line joining the fixed charges, the charge + q executes simple harmonic motion. Find the frequency of oscil-lations.*

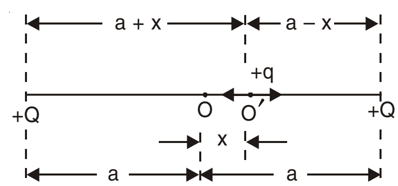

Fig. 1.27

Solution

Let the charge + q be displaced through a distance x to the right (Fig. 1.27). Restoring force on charge + q is

$$F = \frac{Qq}{4\pi \epsilon_0 (a+x)^2} - \frac{Qq}{4\pi \epsilon_0 (a-x)^2}$$

$$= -\frac{4aQq\,x}{4\pi \epsilon_0 (a^2 - x^2)^2}$$

$$\approx -\frac{Qq\,x}{\pi \epsilon_0 a^3} \quad \text{since } x \ll a,$$

where $\epsilon_0 = 8.85 \times 10^{-12}$ C^2/N.m^2.

The equation of motion of the charge + q of mass m *is*

$$m\ddot{x} = -\frac{Qq\,x}{\pi \epsilon_0 a^3}$$

which represents SHM with frequency

$$\nu = \frac{1}{2\pi}\sqrt{\frac{Qq}{\pi \in_0 a^3 m}}.$$

37. *A thin ring of radius R carries uniformly distributed charge + Q. A point charge – q is placed on the axis of the ring at a distance x from the centre of the ring and released from rest. Show that the motion of the charged particle is approximately simple harmonic. Find the frequency of oscillations.*

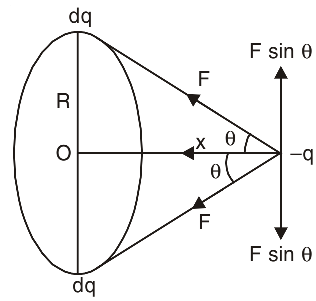

Fig. 1.28

Solution

Consider two symmetric small charge elements dq of the ring. The net force on the point charge along the x-direction (Fig. 1.28) is

$$-\frac{(2dq)q}{4\pi \in_0 \left(R^2 + x^2\right)} \cos\theta = -\frac{(2dq)q\,x}{4\pi \in_0 \left(R^2 + x^2\right)^{3/2}}.$$

Thus the net force on $(-q)$ due to the total charge $(+Q)$ on the ring is

$$F = -\frac{Q\,q\,x}{4\pi \in_0 \left(R^2 + x^2\right)^{3/2}}$$

$$\approx -\frac{Q\,q\,x}{4\pi \in_0 R^3} \quad \text{since } x \ll \text{R}.$$

The equation of motion of the point charge is

$$m\ddot{x} = -\frac{Q\,q\,x}{4\pi \in_0 R^3}$$

which represents SHM with frequency

$$\nu = \frac{1}{2\pi}\sqrt{\frac{Qq}{4\pi \in_0 mR^3}}.$$

38. *An object of 98 N weight suspended from the end of a vertical spring of negligible mass stretches the spring by 0.1 m. (a) Determine the position of the object at any time if initially it is pulled down 0.05 m and then released. (b) Find the amplitude, period and frequency of the motion.*

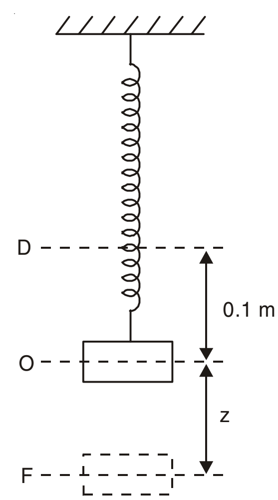

Fig. 1.29

Solution

(*a*) Let D and O represent the position of the end of the spring before and after the object is put (Fig. 1.29). Position O is the equilibrium position of the object. The positive z-axis is downward with origin at the equilibrium position O. When the elongation of the spring is 0.1 m, the force on it is 98 N.

When the elongation is $(0.1 + z)$ m, the force on it is $\frac{98}{0.1} \times (0.1 + z)$ N. Thus, when the object is released at F, there is an upward force acting on it of magnitude $\frac{98}{0.1} \times (0.1 + z)$ N and a downward force due to its weight of magnitude 98 N. Hence, we can write

$$\frac{98}{9.8}\frac{d^2z}{dt^2}\hat{k} = 98\,\hat{k} - \frac{98}{0.1}(0.1+z)\hat{k} = -\frac{98}{0.1}z\hat{k}\cdot$$

The resulting motion is simple harmonic with angular frequency

$$\omega = \sqrt{\frac{9.8}{0.1}} = 7\sqrt{2}\ \text{s}^{-1}\cdot$$

The solution of the differential equation is (see Eqn. 1.6)

$$z = C\cos 7\sqrt{2}\,t + D\sin 7\sqrt{2}\,t\,.$$

At $t = 0$, $z = 0.05$ m, $\frac{dz}{dt} = 0$ which give $C = 0.05$ and $D = 0$. Thus, the position of the object at any time is given by

$$z = 0.05\cos 7\sqrt{2}\,t$$

(b) Amplitude = 0.05 m, period $= \frac{\sqrt{2}\pi}{7}$ s and frequency = $\frac{7\sqrt{2}}{2\pi}$ Hz.

39. *A particle of mass 3 units moves along the x-axis attracted toward origin by a force whose magnitude is numerically equal to 27x. If it starts from rest at x = 8 units, find (a) the differential equation describing the motion of the particle (b) the position and velocity of the particle at any time and (c) The amplitude and period of the vibration.*

Solution

(a) Let $\vec{r} = x\hat{i}$ be the position vector of the particle. The force acting on the particle is

$$3\ddot{x}\hat{i} = -27x\,\hat{i}$$

which gives

$$\ddot{x} + 9x = 0.$$

This is the required differential equation.

(b) The general solution of the differential equation is

$$x = C\cos 3t + D\sin 3t.$$

The initial conditions are $x = 8$, $\dot{x} = 0$ at $t = 0$, which give $C = 8$ and $D = 0$.

Thus,
$$x = 8\cos 3t$$

and the velocity is
$$\frac{dx}{dt}\hat{i} = -24\sin 3t\,\hat{i}$$

(c) Amplitude = 8 units, period = $\frac{2\pi}{3}$ s.

40. *Work the previous problem if the particle is initially at $x = 8$ units but is moving (a) to the right with speed 18 units, (b) to the left with speed 18 units. Find the amplitude, frequency and the phase angle in each case. Are the two motions (a) and (b) 180° out of phase with each other?*

Solution

(*a*) $x = C \cos 3t + D \sin 3t.$

At $t = 0$, $x = 8$ and $\dot{x} = 18$, which give $C = 8$ and $D = 6$.

Thus,
$$x = 8 \cos 3t + 6 \sin 3t$$
$$= \sqrt{8^2 + 6^2} \cos(3t - \phi)$$
$$= 10 \cos(3t - \phi)$$

where
$$\cos \phi = \frac{8}{10}, \sin \phi = \frac{6}{10} \text{ and } \tan \phi = \frac{3}{4}.$$

The angle ϕ is called the phase angle which is in the first quadrant: $\phi = 36.87°$.

Amplitude = 10 and frequency = $\frac{3}{2\pi}$ Hz.

(*b*) At $t = 0$, $x = 8$ and $\dot{x} = -18$, which give $C = 8$ and $D = -6$ so that
$$x = 8 \cos 3t - 6 \sin 3t$$
$$= 10 \cos(3t - \psi)$$

with $\cos \psi = \frac{8}{10}$, $\sin \psi = -\frac{6}{10}$ and $\tan \psi = -\frac{3}{4}$.

The phase angle ψ is in the fourth quadrant:
$$\psi = 323.13°.$$

The amplitude and frequency are the same as in part (*a*). The only difference is in the phase angle. Here we have $\sin(\phi + \psi) = 0$ and $\cos(\phi + \psi) = 1$ and $\psi + \phi = 2\pi$. The two motions are not 180° out of phase with each other since $\psi - \phi \neq 180°$.

41. *A pail of water, at the end of a rope of length r, is whirled in a horizontal circle at constant speed v. A distant ground-level spot light casts a shadow of the pail onto a vertical wall which is perpendicular to the spotlight beam. Show that the shadow executes SHM with angular frequency ω = v/n.*

Solution

The figure gives a top view of the set-up (Fig. 1.30).

Let $\theta(t)$ denote the instantaneous angular position in radians of the pail, measured counter clockwise from the + ve x-axis. Then $\omega = \pm \frac{v}{r}$, with the sign depending upon which way the pail is whirled. Letting $\theta(t = 0) = \theta_0$, the angular position
$$\theta(t) = \pm \frac{v}{r} t + \theta_0.$$

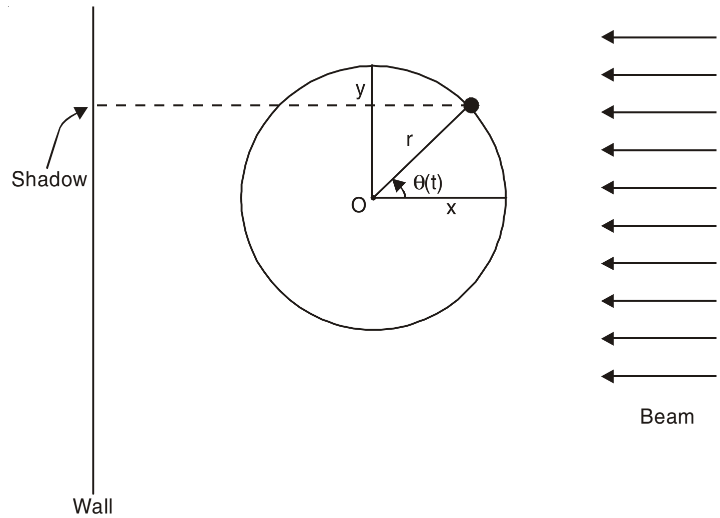

Fig. 1.30

Now, $$y(t) = r \sin \theta(t) = r \sin \left(\pm\frac{vt}{r} + \theta_0\right)$$

$$= \pm r \sin \left(\frac{vt}{r} \pm \theta_0\right)$$

So, $$\ddot{y} = \mp \frac{v^2}{r} \sin \left(\frac{vt}{r} \pm \theta_0\right)$$

and the differential equation for the motion of the shadow is

$$\ddot{y} + \frac{v^2}{r^2} y = 0$$

or $$\ddot{y} + \omega^2 y = 0$$

i.e., we have SHM of amplitude $|\pm r| = r$ and angular frequency $\omega = \frac{v}{r}$.

42. *Two particles oscillate in simple harmonic motion along a common straight line segment of length A. Each particle has a period of 1.5 s but they differ in phase by 30°. (a) How far apart are they (in terms of A) 0.5 s after the lagging particle leaves one end of the path? (b) Are they moving in the same direction, towards each other, or away from each other at this time?*

Solution

(*a*) Let the equations of two particles be

$$x_1 = \frac{A}{2} \cos\left(\frac{4\pi}{3}t + \frac{\pi}{6}\right)$$

$$x_2 = \frac{A}{2} \cos\left(\frac{4\pi}{3}t\right)$$

x_2 reaches one end of the path when $t = 0$ and at that time

$$x_1 = \frac{A}{2} \cdot \frac{\sqrt{3}}{2}.$$

When t = 0.5 s, $x_2 = -\frac{A}{4}$ and $x_1 = -\frac{A\sqrt{3}}{4}$, and $|x_2 - x_1|$ = 0.183 A.

(*b*) At t = 0.5 s, velocities of the particles are

$$\dot{x}_1 = -\frac{A}{2}\frac{4\pi}{3}\sin 150° = -\text{ ve},$$

$$\dot{x}_2 = -\frac{A}{2}\frac{4\pi}{3}\sin 120° = -\text{ ve}.$$

Thus, the particles are moving in the same direction.

43. *Two particles execute SHM of the same amplitude and frequency along the same line. They pass one another when going in opposite directions each time their displacement is half their amplitude. Show that the phase difference between them is 120°.*

Solution

Let the equations be

$$x_1 = A\sin(\omega t + \phi_1),$$
$$x_2 = A\sin(\omega t + \phi_2)$$

with $\phi_1 \neq \phi_2$.

Let the particles cross each other at t = 0, so that

$$x_1 = x_2 = \frac{A}{2} \text{ at } t = 0$$

which give ϕ_1 = 30° and ϕ_2 = 150°.

At t = 0, $\dot{x}_1 = A\omega\cos\phi_1$ = + ve and $\dot{x}_2 = A\omega\cos\phi_2$ = – ve which show that the particles are moving in opposite directions.

Phase difference = $\phi_2 - \phi_1$ = 120°.

44. *Show that the phase-space diagram (p_x versus x curve) of SHM is an ellipse with area equal to $\frac{E}{\nu}$ where E = Total energy and ν = frequency of oscillations.*

Solution

For a particle executing SHM, we have

$$x = A\cos\omega t$$

which gives

$$p_x = m\dot{x} = -m\omega A\sin\omega t$$

and

$$\frac{x^2}{A^2} + \frac{p_x^2}{m^2\omega^2A^2} = 1.$$

Since

$$E = \frac{1}{2}m\omega^2A^2, \text{ we have}$$

$$\frac{x^2}{\frac{2E}{m\omega^2}} + \frac{p_x^2}{2mE} = 1$$

which is the equation of an ellipse in the xp_x–plane with semi-major axis = $a = \sqrt{2E/(m\omega^2)}$ and semi-minor axis = $b = \sqrt{2mE}$.

Area of the ellipse = $\pi ab = \dfrac{2\pi E}{\omega} = \dfrac{E}{\nu}$.

45. *Show that the force* $\vec{F} = -kx\hat{i}$ *acting on a simple harmonic oscillator is conservative.*

Solution

We have
$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -kx & 0 & 0 \end{vmatrix} = 0.$$

Hence $\vec{F}$ is conservative.

46. *Solve the differential equation*

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}kx^2 = E$$

and show that x(t) represents simple harmonic motion. What is the frequency and amplitude of vibration of the motion?

Solution

We have
$$\frac{dx}{dt} = \omega\sqrt{A^2 - x^2}$$

where
$$\omega = \sqrt{\frac{k}{m}} \text{ and } A^2 = \frac{2E}{k}.$$

After integration, we get

$$x = \mathrm{A}\sin(\omega t + \phi)$$

which represents a simple harmonic motion. Here ϕ is the phase angle.

Frequency $= \dfrac{\omega}{2\pi} = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{m}}$ and Amplitude $= A = \sqrt{\dfrac{2E}{k}}$.

47. *A spring of mass M and spring constant k is hanged from a rigid support. A mass m is suspended at the lower end of the spring. If the mass is pulled down and released then it will execute SHM. Find the frequency of oscillations.*

Solution

Let l be the length of the coiled wire of the spring. Let the suspended mass m be at a distance z from the equilibrium position. We consider an element of length dx of the spring situated between points x and $x + dx$ measured along the spring from the point of support. Since the stretching is assumed to be uniform, the element, distant x from the fixed end, will experience a displacement $(x/l)\,z$ and will have velocity $(x/l)\,\dot{z}$. Since the mass of the element dx is $(M/l)\,zdx$, the kinetic energy of this element is

$$\frac{1}{2}\left(\frac{M}{l}\right)dx\left[\frac{xdz}{l\,dt}\right]^2 = \frac{M}{2l^3}\left[\frac{dz}{dt}\right]^2 x^2\,dx.$$

Thus at that instant total kinetic energy of the spring is

$$\frac{M}{2l^3}\left[\frac{dz}{dt}\right]^2 \int_0^1 x^2\, dx = \frac{M}{6}\left(\frac{dz}{dt}\right)^2,$$

and the kinetic energy of the mass m is $\frac{1}{2}m\left(\frac{dz}{dt}\right)^2$.

The potential energy of the system when the spring is elongated by z is $\left(\frac{1}{2}\right)kz^2$.

Hence the total energy of the system is

$$\left(\frac{1}{2}\right)\left(m+\frac{M}{3}\right)\left(\frac{dz}{dt}\right)^2 + \frac{1}{2}kz^2 = \text{constant.} \quad ...(1.19)$$

Due to finite mass of the spring the effective mass of the system becomes $\left(m+\frac{M}{3}\right)$. From problem (46), we find that

$$\omega^2 = \frac{k}{m+\frac{M}{3}} \quad ...(1.20)$$

Differentiating Eqn. (1.19) with respect to t, we obtain

$$\frac{1}{2}\left(m+\frac{M}{3}\right)2\dot{z}\ddot{z} + \frac{1}{2}k\,2z\,\dot{z} = 0$$

or

$$\left(m+\frac{M}{3}\right)\ddot{z} + kz = 0$$

which shows that the motion is simple harmonic with ω^2 given by Eqn. (1.20).

48. *Show that if the assumption of small vibration (see problem 24) is not made, then the time period of a simple pendulum is given by*

$$T = 4\sqrt{\frac{l}{g}}\int_0^{\frac{p}{2}} \frac{df}{\sqrt{1-k^2\sin^2 f}}$$

where $k = \sin\left(\frac{\psi_0}{2}\right)$, ψ_0 *being the maximum angle made by the string with the vertical. The initial conditions are* $\psi = \psi_0$ *and* $\psi = 0$ *at time* $t = 0$.

Solution

The equation of motion for a simple pendulum, if small vibrations are not assumed, is:

$$\frac{d^2\psi}{dt^2} = -\frac{g}{l}\sin\psi \quad ...(1.21)$$

We put $u = \frac{d\psi}{dt}$ so that

$$\frac{d^2\psi}{dt^2} = \frac{du}{dt} = \frac{du}{d\psi}\frac{d\psi}{dt} = u\frac{du}{d\psi}$$

and Eqn. (1.21) becomes

$$u\frac{du}{d\psi} = -\frac{g}{l}\sin\psi.$$

On integration, we get

$$\frac{u^2}{2} = \frac{g}{l}\cos\psi + C.$$

When $\psi = \psi_0$, $\dot{\psi} = u = 0$, so that

$$C = -\frac{g}{l}\cos\psi_0.$$

Thus, we have
$$u = \pm\left[\frac{2g}{l}(\cos\psi - \cos\psi_0)\right]^{\frac{1}{2}}.$$

We consider that part of the motion where the bob goes from $\psi = \psi_0$ to $\psi = 0$ which represents a time equal to $\frac{T}{4}$. In this case ψ is decreasing so that ψ is negative:

$$\frac{d\psi}{dt} = -\sqrt{\frac{2g}{l}}(\cos\psi - \cos\psi_0)^{\frac{1}{2}}.$$

Integrating from $\psi = \psi_0$ to $\psi = 0$, we get

$$\frac{T}{4} = -\sqrt{\frac{l}{2g}}\int_{\psi_0}^{0}\frac{d\psi}{(\cos\psi - \cos\psi_0)^{1/2}}$$

$$= \sqrt{\frac{l}{2g}}\int_{0}^{\psi_0}\frac{d\psi}{\sqrt{2}\left(\sin^2\frac{\psi_0}{2} - \sin^2\frac{\psi}{2}\right)^{1/2}}.$$

Let $\sin\frac{\psi}{2} = \sin\frac{\psi_0}{2}\sin\phi$, so that

$$\frac{1}{2}\cos\frac{\psi}{2}\,d\psi = \sin\frac{\psi_0}{2}\cos\phi\,d\phi.$$

Putting $k = \sin\frac{\psi_0}{2}$, we get

$$T = 4\sqrt{\frac{l}{g}}\int_{0}^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} \quad \text{...(1.22)}$$

This integral is called an elliptical integral. For small vibrations $k \approx 0$ and

$$T = 2\pi\sqrt{\frac{l}{g}}.$$

49. *Show that the time period given in problem 48 can be written as*

$$T = 2p\sqrt{\frac{l}{g}}\left[1+\left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1.3}{2.4}\right)^2 k^4 + ...\right]$$

Solution

Since $k^2 \sin^2 \phi < 1$, we make binomial expansion of $\left(1 - k^2 \sin^2 \phi\right)^{-1/2}$ and integrate term by term. We finally find

$$T = 2\pi\sqrt{\frac{l}{g}}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1.3}{2.4}\right)^2 k^4 + ...\right]$$

where we have made use of the following integration formula

$$\int_0^{\pi/2} \sin^{2n} \phi d\phi = \frac{1.3.5....(2n-1)}{2.4.6....2n}\frac{\pi}{2}.$$

50. *A particle of mass m is located in a one-dimensional potential field where the potential energy of the particle depends on the coordinates x as V(x) = V_0 (1 – cos ax). Find the period of small oscillations that the particle performs about the equilibrium position.*

Solution

$$\frac{dV}{dx} = 0 \text{ when } \sin ax = 0 \text{ or } x = 0.$$

Again $x = 0$ is the point of minimum of $V(x)$ since $\frac{d^2V}{dx^2} > 0$ at $x = 0$. The force acting on the particle is

$$F_x = -\frac{\partial V}{\partial x} = -V_0 a \sin ax.$$

For small values of x we have

$$m\ddot{x} = -V_0 a(ax) = -V_0 a^2 x.$$

The time period of small oscillations is

$$T = \frac{2\pi}{a}\sqrt{\frac{m}{V_0}}.$$

51. *A bead of mass m slides on a frictionless wire of nearly parabolic shape (Fig. 1.31). Let the point P be the point at the bottom of the wire. Show that the bead will oscillate about P if displaced slightly from P and released.*

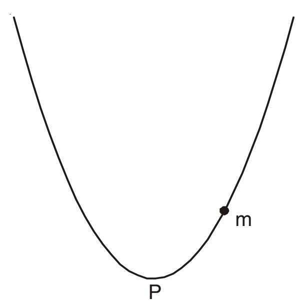

Fig. 1.31

Solution

Since the shape of the wire near P is a parabola, the potential energy of the bead is given by $V = cx^2$ in the neighbourhood of P, where x is measured from P and c is a constant. Now, $F_x = -\frac{\partial V}{\partial x} = -2cx$, *i.e.*, $F_x \propto x$ and directed opposite to the direction of increasing x. So, the bead executes SHM and its time period is given by $T = 2\pi\left(\frac{m}{2c}\right)^{1/2}$.

Note that this line of reasoning leads to a general result: Any conservative system will oscillate with SHM about a minimum in its potential energy curve provided the oscillation amplitude is small enough.

52. *The potential energy of a particle of mass m is given by*

$$V(x) = \frac{a}{x^2} - \frac{b}{x}$$

where a and b are positive constants. Find the minimum of V(x) and expand V(x) about the point of minimum of V(x). Find the period of small oscillations that the particle performs about the position of minimum of V(x).

Solution

$\frac{dV}{dx} = 0$ when $x = \frac{2a}{b}$ and $\frac{d^2V}{dx^2} > 0$ at $x = \frac{2a}{b}$. Hence $x = \frac{2a}{b}$ is the point of minimum of $V(x)$.

Now,
$$V\left(\frac{2a}{b}\right) = -\frac{b^2}{4a}$$
$$V'\left(\frac{2a}{b}\right) = 0$$
$$V''\left(\frac{2a}{b}\right) = \frac{b^4}{8a^3}$$

Thus the expansion of $V(x)$ about $x = \frac{2a}{b}$ is given by

$$V(x) = -\frac{b^2}{4a} + \frac{\left(x - \frac{2a}{b}\right)^2}{2!}\frac{b^4}{8a^3} + \cdots$$

If we put $y = x - \left(\frac{2a}{b}\right)$, the equation of motion about the position of minimum of $V(x)$ is

$$m\ddot{y} = -\frac{b^4}{8a^3}y.$$

Time period
$$T = 2\pi/\left(\frac{b^4}{8a^3 m}\right)^{\frac{1}{2}} = \frac{4\pi a}{b^2}\sqrt{2ma}.$$

53. *A thin rod of length 10 cm and mass 100 g is suspended at its midpoint from a long wire. Its period T_a of angular SHM is measured to be 2 s. An irregular object, which we call object X, is then hung from the same wire, and its period T_x is found be 3 s. What is the rotational inertia of the object X about its suspension axis?*

Solution

The moment of inertia of the thin rod about a perpendicular axis through its midpoint is

$$I_a = \frac{1}{12}mL^2 = \frac{1}{12} \times 0.1 \times (0.1)^2 = \frac{1}{12} \times 10^{-3} \text{ kg.m}^2$$

We know $$T_a = 2\pi\sqrt{\frac{I_a}{C}} \text{ and } T_x = 2\pi\sqrt{\frac{I_x}{C}}$$

Thus, $$I_x = I_a \frac{T_x^{\ 2}}{T_a^2} = \frac{1}{12} \times 10^{-3} \times \frac{9}{4}$$
$$= 1.875 \times 10^{-4} \text{ kg.m}^2.$$

54. *A uniform disc of radius R and mass M is attached to the end of a uniform rigid rod of length L and mass m. When the disc is suspended from a pivot as shown in Fig. 1.32, what will be the period of motion?*

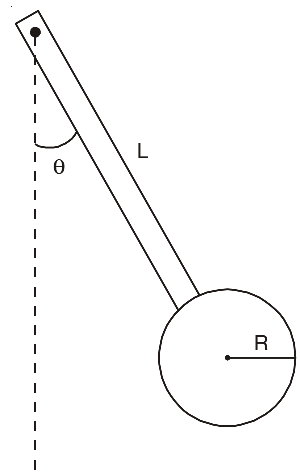

Fig. 1.32

Solution

The equation of motion is $\Gamma = I\alpha$, where Γ is the external torque, I is the moment of inertia, α is the angular acceleration, and both Γ and I are about the pivot point. Let θ be a small angular displacement from the vertical. Now, external torque comes both from the rod and the disc:

$$\Gamma = -mg\frac{L}{2}\sin\theta - Mg(R+L)\sin\theta$$
$$\approx -\left[\frac{1}{2}mgL + Mg(R+L)\right]\theta = -C\theta$$

Where we put $\sin\theta \approx \theta$.

Now, $$I = I_{\text{rod}} + I_{\text{disc}} = \frac{1}{3}mL^2 + \left[\frac{1}{2}MR^2 + M(R+L)^2\right].$$

The time period T is given by

$$T = 2\pi\sqrt{\frac{I}{C}}$$

$$= 2\pi\left(\frac{L}{g}\right)^{\frac{1}{2}} \frac{\left[\frac{a}{3} + \frac{b^2}{2} + (1+b)^2\right]^{1/2}}{\left[1 + b + \frac{a}{2}\right]^{1/2}}$$

Where $$a = \frac{m}{M} \text{ and } b = \frac{R}{L}.$$

55. *A thin rod of length L and area of cross-section S is pivoted at its lowest point P inside a stationary, homogeneous and non-viscous liquid (Fig. 1.33). The rod is free to rotate in a vertical plane about a horizontal axis passing through P. The density d_1 of the material of the rod is smaller than the density d_2 of the liquid. The rod is displaced by a small angle θ from its equilibrium position and then released. Show that the motion of the rod is simple harmonic and determine its angular frequency in terms of the given parameters.* (I.I.T. 1996)

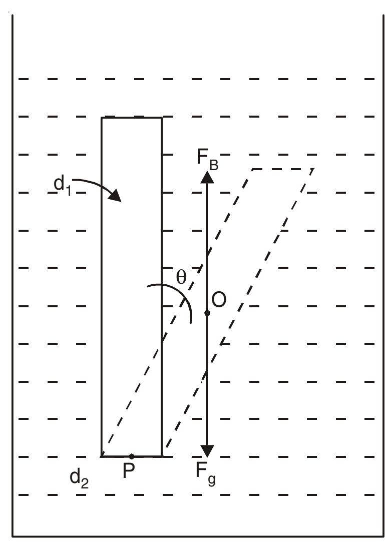

Fig 1.33

Solution

Volume of the rod, $V = SL$ and its mass, $M = Vd_1 = SLd_1$. In the slightly displaced position two forces are acting on the rod:

Downward force due to gravity, $F_g = Mg = SLd_1 g$.

Upward force due to buoyancy, $F_B = Vd_2 g = SLd_2 g$.

Since $d_2 > d_1$, a net upward force of magnitude $F_B - F_g$ acts at O (the middle point of the rod).

Torque due to the net upward force about P is

$$\Gamma = -SLg(d_2 - d_1)\frac{L}{2}\sin\theta$$

$$\approx -\frac{1}{2}SL^2 g(d_2 - d_1)\theta = -C\theta.$$

The negative sign is due to the fact that the torque acts in the opposite direction of increasing θ. Now, the moment of inertia of the rod about the pivot point P is

$$I = \frac{1}{3}ML^2 = \frac{1}{3}Sd_1 L^3.$$

The equation of motion of the rod is

$$\Gamma = I\frac{d^2\theta}{dt^2}$$

or

$$\frac{d^2\theta}{dt^2} = -\frac{C}{I}\theta = -\omega^2\theta.$$

The motion of the rod is simple harmonic. The angular frequency is

$$\omega = \sqrt{\frac{C}{I}} = \left[\frac{3g(d_2 - d_1)}{2d_1 L}\right]^{1/2}.$$

56. *A thin light beam of uniform cross-section A is clamped at one end and loaded at free end by placing a mass M. [Such a beam is called a loaded cantilever.] If the loaded free end of the beam is slightly displaced from its equilibrium position, it starts executing SHM. Find an expression for the time period of vibration of the loaded light cantilever.*

Solution

We shall assume that the bar is not subjected to any tension and the amplitude of motion is so small that the rotatory effect can be neglected. The x-axis is taken along the length of the bar and the transverse vibration is taking place in the y-direction. The radius of curvature R is given by

$$\frac{1}{R} = \frac{d^2y}{dx^2}\Bigg/\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}.$$

For a small transverse vibration $\dfrac{dy}{dx} \ll 1$ and

$$\frac{1}{R} \approx \frac{d^2y}{dx^2}.$$

In the bent position of the rod we consider a cross-section ABCD of a small segment of the rod of length δx measured along the central line PQ so that $EQ = EP = R$ = radius of curvature of the central filament PQ (Fig. 1.34). The filaments above PQ are extended whereas the filaments below PQ are contracted. PQ is the neutral filament and $PQ = \delta x$. Let us consider a filament MN above PQ at a distance r from PQ. Let Δ be the extension of this filament so that $MN = \delta x + \Delta$. From Fig. 1.34 we have

$$\phi = \frac{\delta x}{R} = \frac{\delta x + \Delta}{R + r}$$

or

$$\Delta = \frac{r\delta x}{R}.$$

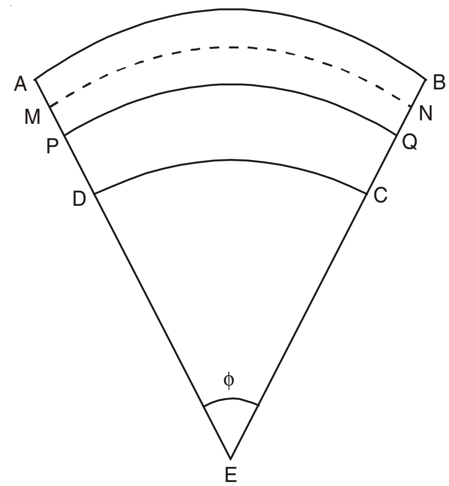

Fig. 1.34

Thus, $$\text{longitudinal strain} = \frac{\Delta}{\delta x} = \frac{r}{R}.$$

If Y denotes the Young's modulus of the material of the beam, the longitudinal stress is $\frac{Yr}{R}$. Hence the force acting on the filament is $\alpha \frac{Yr}{R}$, where α is the area of cross-section of the filament MN. The total bending moment of the bar is

$$\Gamma = \frac{Y}{R} \Sigma r^2 \alpha \approx Y \frac{d^2 y}{\delta x^2} \cdot I_g$$

where $\Sigma r^2 \alpha = AK^2 = I_g$ is known as geometrical moment of inertia of the cross-section of the rod about the neutral axis, and K is the radius of gyration of the section about the neutral axis.

Let OG be the cantilever of length l clamped at the end O and loaded at the free end G (Fig. 1.35). We neglect the weight of the beam. We are interested in finding the depression at any point F (x, y) of the cantilever. The bending moment at F is

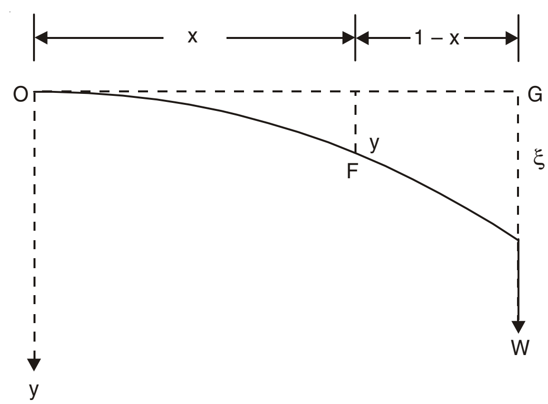

Fig. 1.35

$W(l - x)$ where $W = Mg$. This equals the resisting moment $YI_g \dfrac{d^2y}{dx^2}$. Thus the differential equation for the bending of the beam is

$$\frac{d^2y}{dx^2} = \frac{W}{YI_g}(l - x).$$

Integrating, we get
$$\frac{dy}{dx} = \frac{W}{YI_g}(lx - x^2/2) + C_1$$

where C_1 is the constant of integration.

Now, $\dfrac{dy}{dx} = 0$ when $x = 0$ so that $C_1 = 0$.

Integrating again, we get

$$y = \frac{W}{YI_g}(lx^2/2 - x^3/6) + C_2.$$

We have the boundary condition, $y = 0$ when $x = 0$ so that $C_2 = 0$. The depression of the loaded end $(x = l)$ is

$$\xi = \frac{W}{YI_g}\left(\frac{l^3}{3}\right) = \frac{l^3}{3YI_g}(Mg)$$

or
$$Mg = \frac{3YI_g}{l^3}\xi.$$

The restoring force in magnitude on the mass M is $\left(\dfrac{3YI_g}{l^3}\right)\xi$ which is proportional to ξ. The restoring force is opposite to the direction of increasing ξ. The mass M executes SHM which may be written as

$$M\frac{d^2\xi}{dt^2} + \frac{3YI_g}{l^3}\xi = 0.$$

The time period of vibration of the mass is

$$T = 2\pi\left[\frac{Ml^3}{3YI_g}\right]^{1/2}.$$

57. *A rectangular light beam of breadth b, thickness d and length l is clamped at one end and loaded at free end by placing a mass M. Show that the time period of vibration of the mass is*

$$T = 2\pi\left[\frac{4Ml^3}{Ybd^3}\right]^{1/2}$$

Solution

Let us consider the cross-section of the beam (Fig. 1.36). *PQ* is the neutral line. We consider the strip *ST* of thickness dx at a distance x from the neutral axis *PQ*. The geometrical moment of inertia of the cross-section of the beam about the neutral axis is

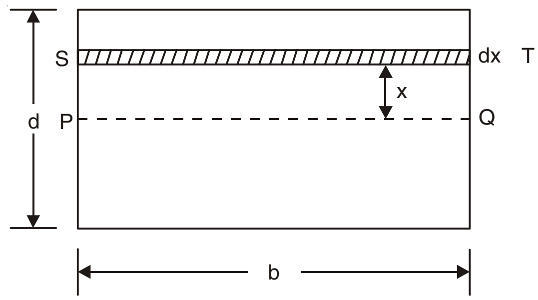

Fig. 1.36

$$I_g = 2\int_0^{d/2} bdx \cdot x^2 = \frac{bd^3}{12}.$$

Thus, $$T = 2\pi\left[\frac{4Ml^3}{Ybd^3}\right]^{1/2}.$$

58. *A light beam of circular cross-section of radius a and length l is clamped at one end and loaded at free end by placing a mass M. Show that the time period of vibration of this rod is*

$$T = \frac{2\pi}{a^2}\left[\frac{4Ml^3}{3\pi Y}\right]^{1/2}.$$

Solution

We consider an elementary area $dr\ rd\theta$ of the circular cross-section of the rod (Fig. 1.37). The geometrical moment of inertia of the cross-section of the rod about the neutral line *PQ* is

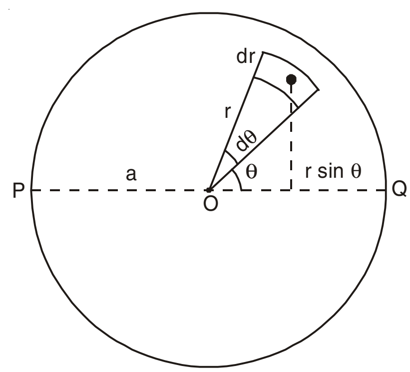

Fig. 1.37

$$I_g = 2\int_0^a \int_0^\pi (r\,dr\,d\theta)(r\sin\theta)^2 = \frac{\pi a^4}{4}.$$

Hence, $$T = \frac{2\pi}{a^2}\left[\frac{4Ml^3}{3\pi Y}\right]^{1/2}.$$

59. *We would like to make an LC circuit that oscillates at 440 Hz. If we have a 2 H inductor, what value of capacitance should we use? If the capacitor is initially charged to 5 V, what will be the peak charge on the capacitor? What is the total energy in the circuit ?*

Solution

The total energy in the circuit is the sum of the magnetic and electric energy:

$$E = E_B + E_E = \frac{1}{2}LI^2 + \frac{1}{2}\frac{q^2}{C}$$

where I = Current and q = Capacitor charge.

Since the total energy does not change, $\frac{dE}{dt} = 0$. Thus we have

$$LI\frac{dI}{dt} + \frac{q}{C}\frac{dq}{dt} = 0.$$

Substituting $I = \dfrac{dq}{dt}$ and $\dfrac{dI}{dt} = \dfrac{d^2q}{dt^2}$, we obtain

$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = 0$$

which describes the capacitor charge as a function of time. The solution of this equation is

$$q = q_0 \cos(\omega t + \phi)$$

where $\omega = \dfrac{1}{\sqrt{LC}}$ is the oscillation angular frequency. Thus,

$$C = \frac{1}{\omega^2 L} = \frac{1}{4\pi^2\nu^2 L} = 0.065\ \mu\text{F}.$$

Peak charge, $q_0 = CV_0 = (0.065\ \mu\text{F})\ (5.0\ \text{V})$
$= 0.33\ \mu\text{C}.$

Now, $I = \dfrac{dq}{dt} = -\omega q_0 \sin(\omega t + \phi)$

Peak current, $I_0 = \omega q_0 = 2\pi\nu q_0 = 0.91$ m A

$$\text{Total energy} = \frac{1}{2}\frac{q_0^2}{C} = \frac{0.33 \times 0.33}{2 \times 0.065}\ \mu\text{J}$$
$$= 0.84\ \mu\text{J}.$$

60. *Two particles of mass m each are tied at the ends of a light string of length 2a. The whole system is kept on a frictionless horizontal surface with the string held tight so that each mass is at a distance a from the centre P (as shown in the Figure 1.38). Now the mid-point of the string is pulled vertically upwards with a small but constant force F. As a result, the particles move towards each other on the surface.*

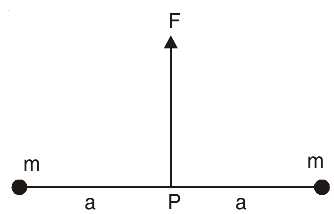

Fig. 1.38

The magnitude of acceleration, when the separation between them becomes 2x, is

(*a*) $\dfrac{F}{2m}\dfrac{a}{\sqrt{a^2 - x^2}}$ (*b*) $\dfrac{F}{2m}\dfrac{x}{\sqrt{a^2 - x^2}}$

(*c*) $\dfrac{F}{2m}\dfrac{x}{a}$ (*d*) $\dfrac{F}{2m}\dfrac{\sqrt{a^2 - x^2}}{x}$ (I.I.T. 2007)

Solution

From Fig. 1.39, we have

$$F = 2T \sin\theta$$

and $T\cos\theta = mf$

where f is the acceleration

Thus, $$f = \frac{T\cos\theta}{m} = \frac{F}{2\sin\theta}\cdot\frac{\cos\theta}{m}$$
$$= \frac{F}{2m}\frac{x}{\sqrt{a^2 - x^2}}$$

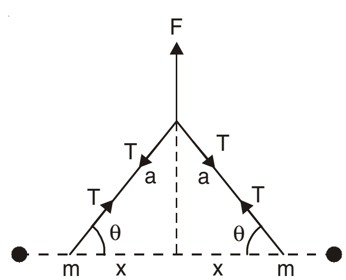

Fig. 1.39

Correct choice : (*b*)

61. *A student performs an experiment for determination of g* $\left(=\dfrac{4\pi^2 l}{T^2}\right)$, $l \approx 1m$, *and he commits an error of* Δl. *For T, he takes the time of n oscillations with the stop watch of least count* ΔT *and he commits a human error of 0.1 s. For which of the following data, the measurement of g will be most accurate?*

	Δl	ΔT	n	*Amplitude of oscillation*
(*a*)	*5 mm*	*0.2s*	*10*	*5 mm*
(*b*)	*5 mm*	*0.2s*	*20*	*5 mm*
(*c*)	*5 mm*	*0.1s*	*20*	*1 mm*
(*d*)	*1 mm*	*0.1s*	*50*	*1 mm*

(I.I.T. 2006)

Solution

The error in T decreases with increase in the number of oscillations (n). The amplitude should be small for SHM of the simple pendulum. In (D), we have minimum error in l (Δl = 1 mm) and T (ΔT = 0.1s).

Correct choice: (*d*).

62. (*a*) *A small body attached to one end of vertically hanging spring is performing SHM about its mean position with angular frequency* ω *and amplitude a. If at a height* y^* *from the mean position the body gets detached from the spring, calculate the value of* y^* *so that the height H attained by the mass is maximum. The body does not interact with the spring during its subsequent motion after the detachment* ($a\omega^2 > g$).

(*b*) *Find the maximum value of H.* (I.I.T. 2005)

Solution

(*a*) The spring is elongated by a distance l due to the weight mg. Thus, we have

$$kl = mg \text{ or, } l = \frac{mg}{k} = \frac{g}{\omega^2} < a$$

where k is the spring constant and $\omega^2 = (k/m)$ The amplitude of oscillation a is greater than l. Now if the mass is pulled down through a distance from the equilibrium position A (Fig. 1.40) and released from rest it executes SHM about the mean position. When the mass is moving up, suppose, it is at the position C at a distance y^* from the mean position. At this position P.E. stored in the spring $= \dfrac{1}{2}k\,(y^* - l)^2$

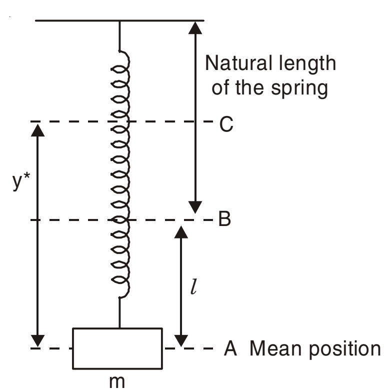

Fig. 1.40

Gravitational P.E. = mgy^*

[Zero of Gravitational P.E. is taken at level A (mean position]

Total energy of the system is

$$E = \frac{1}{2}k\,(y^* - l)^2 + mgy^* + \text{K.E. of the mass at } C = \text{Constant}$$

when $\quad y^* = a$, K.E. of the mass = 0

Thus,
$$E = \frac{1}{2}k\,(a - l)^2 + mga$$
$$= \frac{1}{2}k\,(a^2 + l^2).$$

Suppose, when the mass is moving up at C, it gets detached from the spring, and due to its K.E. It goes further up by a height h so that the K.E. of the mass at $C = mgh$.

Thus, we have
$$E = \frac{1}{2}k\,(y^* - l)^2 + mgy^* + mgh = \frac{1}{2}k\,(a^2 + l^2)$$
or
$$h = \left[\frac{1}{2}k\left(a^2 + l^2\right) - \frac{1}{2}k\left(y^* - l\right)^2 - mgy^*\right]\Big/mg$$

We have to find a condition so that $y^* + h = H$ is maximum.
$$H = y^* + h = \left[\frac{1}{2}k\left(a^2 + l^2\right) - \frac{1}{2}k\left(y^* - l\right)^2\right]\Big/mg$$
$$\frac{dH}{dy^*} = -\frac{k\,(y^* - l)}{mg} = 0$$
or
$$y^* = l = \frac{mg}{k} = \frac{g}{\omega^2}$$
and
$$\frac{d^2H}{dy^{*2}} = -\frac{k}{mg} = -\text{ve}$$

Thus H attains its maximum value when $y^* = l$ [at the position B]. The spring has its natural length at this position.

(b) The maximum value of H is
$$H_{\max} = \frac{1}{2}k\left(a^2 + \frac{m^2g^2}{k^2}\right)\Big/mg$$
$$= \frac{1}{2}\frac{ka^2}{mg} + \frac{1}{2}\frac{mg}{k}$$
$$= \frac{1}{2}\frac{\omega^2a^2}{g} + \frac{1}{2}\frac{g}{\omega^2}.$$

63. *A solid sphere of radius R is floating in a liquid of density ρ with half of its volume submerged. If the sphere is slightly pushed and released, it starts performing simple harmonic motion. Find the frequency of these oscillations.* (I.I.T. 2004)

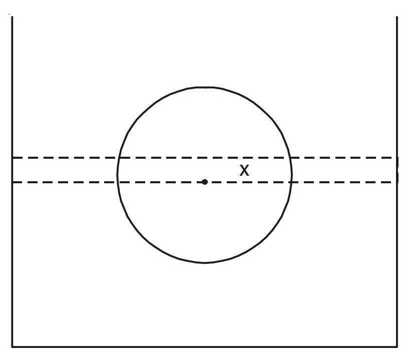

Fig. 1.41

Solution

Initially at equilibrium, mass of the solid sphere = Mass of the displaced liquid

or
$$\frac{4}{3}\pi R^3\rho_1 = \frac{1}{2}\left(\frac{4}{3}\pi R^3\right)\rho$$

where ρ_1 = Density of the solid sphere

Thus, we have $2\rho_1 = \rho$.

Now, the sphere is pushed downward slightly by a distance x inside the liquid (Fig. 1.41).

Net downward force on the sphere is

$$\frac{4}{3}\pi R^3\rho_1 g - \left(\frac{1}{2}\cdot\frac{4}{3}\pi R^3 + \pi R^2 x\right)\rho g = -\pi R^2 x\rho g$$

Thus, the restoring force $= -\pi R^2 \rho g x = m\ddot{x}$

or $$\ddot{x} = -\frac{\pi^2 R^2 \rho g}{m}x = -\omega^2 x.$$

The motion is simple harmonic with

$$\omega^2 = \frac{\pi R^2 \rho g}{\frac{4}{3}\pi R^3 \rho_1} = \frac{3}{2}\frac{g}{R}$$

The frequency of oscillation is

$$\nu = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{3g}{2R}}.$$

64. *A particle of mass m moves on the x-axis as follows: it starts from rest at t = 0 from the point x = 0 and comes to rest at t = 1 at the point x = 1. No other information is available about its motion at intermediate times (0 < t < 1). If α denotes the instantaneous acceleration of the particle, then*

(*a*) *α cannot remain positive for all t in the interval (0 ≤ t ≤ 1).*

(*b*) *|α| cannot exceed 2 at any point in its path.*

(*c*) *|α| must be ≥ 4 at some point or points in its path.*

(*d*) *α must change sign during the motion, but no other assertion can be made with the information given.* (I.I.T. 1993)

Solution

We may consider a motion of the type

$$x = x_0 + A\cos\omega t$$

so that $$\dot{x} = -A\omega\sin\omega t = 0 \text{ at time } t = 0$$

Again, $$\dot{x} = 0 \text{ at time } t = 1$$

or $$\sin\omega = 0 \text{ or } \omega = \pi, 2\pi, 3\pi, \ldots$$

ω cannot be zero. In that case x becomes independent of t. Thus the equation of motion of the particle is

$$x = x_0 + A\cos n\pi t,\ n = 1, 2, 3, \ldots$$

At $t = 0$, $x = 0$ and $t = 1$, $x = 1$,

$$x_0 = -A$$

and $$1 = x_0(1 - \cos n\pi)$$

or $$x_0 = \frac{1}{1-\cos n\pi},\ n \neq 2, 4, 6, \ldots..$$

$$n = 1, 3, 5, \ldots$$

or $$x_0 = \frac{1}{2}$$

Thus, the equation of the particle satisfying all the condition is

$$x = \frac{1}{2}(1 - \cos n\pi t),\ n = 1, 3, 5$$

Acceleration $$\alpha = \ddot{x} = \frac{1}{2}(n\pi)^2 \cos n\pi t$$

$\cos n\pi t$ changes sign when t varies from 0 to 1.

Maximum value of $|\alpha|$ is $\frac{n^2\pi^2}{2} > 4$.

Correct choice : (*a*) and (*c*).

65. *A spring of force constant k is cut into two pieces such that one piece is n times the length of the other. Find the force constant of the long piece.*

Solution

If the spring is divided into $(n + 1)$ equal parts then each has a spring constant $(n + 1)\ k$. The long piece has n such springs which are in series. The equivalent spring constant K of the long piece is given by

$$\frac{1}{K} = \frac{1}{(n+1)k} + \frac{1}{(n+1)k} + \text{.........} n \text{ terms}$$

$$= \frac{n}{(n+1)k}$$

or $$K = \frac{(n+1)\,k}{n}.$$

66. *A particle free to move along the x-axis has potential energy given by*

$$U(x) = k\,[1 - \exp(-x^2)],\ -\alpha \le x \le \alpha$$

where k is a positive constant dimensions. Then

(*a*) *At points away from the origin, the particle is in unstable equilibrium.*

(*b*) *For any finite non-zero value of x, there is a force directed away from the origin.*

(*c*) *If its total mechanical energy is* $k/2$, *it has its minimum kinetic energy at the origin.*

(*d*) *For small displacement from x = 0, the motion is simple harmonic.* (I.I.T. 1999)

Solution

$$\text{Force} = -\frac{dU}{dx} = -\ 2kx\ e^{-x^2}$$

The –ve sign indicates that the force is directed towards the origin.

For small x, Force $= -\ 2kx$, the motion is simple harmonic.

For small x, $U(x) \approx k\ [1 - 1 + x^2] = kx^2$.

Minimum P.E. is at $x = 0$ and thus the maximum K.E. is at $x = 0$.

Far away from the origin $U(x) \approx k$ and the force $= -\frac{dU}{dx} = 0$ [stable equilibrium]

Correct choice : (*d*).

67. *A particle of mass m is executing oscillations about the origin on the x-axis. Its potential energy $U(x) = kx^3$ where k is a positive constant. If amplitude of oscillation is a, then its time period T is*

(*a*) *Proportional to* $\dfrac{1}{\sqrt{a}}$ (*b*) *Independent of a*

(*c*) *Proportional to* $\sqrt{a}$ (*d*) *Proportional to* $a^{3/2}$. (I.I.T. 1998)

Solution

For $x > 0$

Total energy $= E = \dfrac{1}{2}mv^2 + kx^3 = ka^3$ from conservation of energy.

Thus,
$$v = \pm \left(\frac{2k}{m}\right)^{1/2} \sqrt{a^3 - x^3} = \frac{dx}{dt}$$

or
$$dt = \left(\frac{m}{2k}\right)^{1/2} \frac{dx}{\sqrt{a^3 - x^3}}.$$

We consider +ve velocity.

Integrating from $x = 0$ to $x = a$, we have
$$\int_0^{T/4} dt = \left(\frac{m}{2k}\right)^{1/2} \int_0^a \frac{dx}{\sqrt{a^3 - x^3}}.$$

We put $x = a \sin \theta$ so that $dx = a \cos \theta \, d\theta$

Thus,
$$\frac{T}{4} = \left(\frac{m}{2k}\right)^{1/2} \int_0^{\pi/2} \frac{a \cos\theta \, d\theta}{\sqrt{a^3 - a^3 \sin^3\theta}}$$
$$= \left(\frac{m}{2k}\right)^{1/2} \frac{a}{a^{3/2}} \int_0^{\pi/2} \frac{\cos\theta \, d\theta}{\sqrt{1 - \sin^3\theta}}$$

The integral is a constant
$$T = \text{Const.} \frac{1}{\sqrt{a}}$$

or
$$T \propto \frac{1}{\sqrt{a}}$$

Correct choice : (*a*).

68. *Two blocks A and B each of mass m are connected by a massless spring of natural length L and spring constant K. The blocks are initially resting on a smooth horizontal floor with the spring at its natural length as shown in Fig. 1.42.*

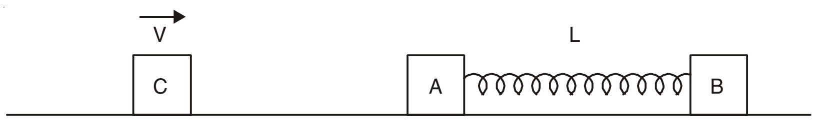

Fig. 1.42

A third identical block C also of mass m, moves on the floor with a speed V along the line joining A and B and collides with A. Then

(a) *the kinetic energy of the A–B system at maximum compression of the spring, is zero,*

(b) *the kinetic energy of the A–B system at maximum compression of the spring is* $mV^2/4$.

(c) *the maximum compression of the spring is* $V\sqrt{m/K}$.

(d) *the maximum compression of the spring is* $V\sqrt{m/2K}$. (I.I.T. 1993)

Solution

The block C will come to rest after colliding with the block A and its energy will be partly converted to the K.E. of the A-B system and the remaining energy goes into the internal energy of the A-B system.

Suppose, V' = Velocity of the A-B system after the collision. From the principle of conservation of momentum, we get

$$mV = 2mV' \quad \text{or,} \quad V' = \frac{V}{2}$$

At the maximum compression of the spring the internal energy is the potential energy of the spring. The A-B system moves with velocity V' after the collision. Thus, the kinetic energy of the A-B system is

$$\frac{1}{2}(2m)V'^2 = m\left(\frac{V}{2}\right)^2 = \frac{mV^2}{4}$$

The P.E. of the A-B system is

$$\text{P.E.} = \frac{1}{2}mV^2 - \frac{mV^2}{4} = \frac{mV^2}{4}$$

If x is the maximum compression of the spring, then

$$\text{P.E.} = \frac{1}{2}Kx^2 = \frac{mV^2}{4}$$

or

$$x = V\sqrt{\frac{m}{2K}}$$

Correct Choice : (b) and (d).

SUPPLEMENTARY PROBLEMS

1. A point moves with SHM. When the point is at 3 cm and 4 cm from the centre of its path, its velocities are 8 cm/s and 6 cm/s respectively. Find its amplitude and time period. Find its acceleration when it is at the greatest distance from the centre.

2. A particle is moving with SHM in a straight line. When the distances of the particle from the equilibrium position are x_1 and x_2, the corresponding values of the velocity are u_1 and u_2. Show that the period is

$$T = 2\pi\left[\left(x_2^2 - x_1^2\right)/\left(u_1^2 - u_2^2\right)\right]^{\frac{1}{2}}.$$

3. A particle of mass 0.005 kg is vibrating 15 times per second with an amplitude of 0.08 m. Find the maximum velocity and its total energy.

4. A particle moves with SHM. If its acceleration at a distance d from the mean position is a, show that the time period of motion is $2\pi\sqrt{d/a}$.

5. At the moment $t = 0$ a body starts oscillating along the x-axis according to the law

$$x = A \sin \omega t.$$

Find (*a*) the mean value of its velocity $<v>$ and (*b*) the mean value of the modulus of the velocity $<|v|>$ averaged over 3/8 of the period after the start.

6. Plot (dP/dx) of problem 6 (page 9) as a function of x. Find the probability of finding the particle within the interval from $-(A/2)$ to $+(A/2)$.

7. A particle is executing SHM. Show that, average K.E. over a cycle = average P.E. over a cycle = Half of the total energy.

8. A particle moves with simple harmonic motion in a straight line. Its maximum speed is 4 m/s and its maximum acceleration is 16 m/s^2. Find (*a*) the time period of the motion, (*b*) the amplitude of the motion.

9. A loudspeaker produces a musical sound by the oscillation of a diaphragm. If the amplitude of oscillation is limited to 9.8×10^{-4} mm, what frequency will result in the acceleration of the diaphragm exceeding g?

10. A small body is undergoing SHM of amplitude A. While going to the right from the equilibrium position, it takes 0.5 s to move from $x = +(A/2)$ to $x = +A$. Find the period of the motion.

11. A block is on a piston that is moving vertically with SHM. (*a*) At what amplitude of motion will the block and piston separate if time period = 1 s? (*b*) If the piston has an amplitude of 4.0 cm, what is the maximum frequency for which the block and piston will be in contact continuously?

12. The piston in the cylindrical head of a locomotive has a stroke of 0.8 m. What is the maximum speed of the piston if the drive wheels make 180 rev/min and the piston moves with simple harmonic motion?

$$\left[\textbf{Hints: } \nu = \frac{180}{60} = 3 \text{ Hz and } A = 0.4 \text{ m}\right].$$

13. A 40 g mass hangs at the end of a spring. When 25 g more is added to the end of the spring, it stretches 7.0 cm more. (*a*) Find the spring constant and (*b*) if 25 g is now removed, what will be the time period of the motion?

14. Two bodies *M* and *N* of equal masses are suspended from two separate massless springs of spring constants k_1 and k_2 respectively. If the two bodies oscillate vertically such that their maximum velocities are equal, the ratio of the amplitude of *M* to that of *N* is

(*a*) $\dfrac{k_1}{k_2}$ (*b*) $\sqrt{\left(\dfrac{k_1}{k_2}\right)}$ (*c*) $\dfrac{k_2}{k_1}$ (*d*) $\sqrt{\left(\dfrac{k_2}{k_1}\right)}$ (*I.I.T. 1988*)

15. A block whose mass is 700g is fastened to a spring whose spring constant *k* is 63 N/m. The block is pulled a distance 10 cm from its equilibrium position and released from rest. (*a*) Find the time period of oscillation of the block, (*b*) what is the mechanical energy of the oscillator? (*c*) What are the potential energy and kinetic energy of this oscillator when the particle is halfway to its end point? [Neglect gravitational P.E.]

16. A cubical block vibrates horizontally in SHM with an amplitude of 4.9 cm and a frequency of 2 Hz. If a smaller block sitting on it is not to slide, what is the minimum value that the coefficient of static friction between the two blocks can have?

[**Hints:** Maximum force on the smaller body = $m\omega^2 A = \mu mg$]

17. The vibration frequencies of atoms in solids at normal temperatures are of the order of 10^{13} Hz. Imagine the atoms to be connected to one another by "springs". Suppose that a single silver atom vibrates with this frequency and that all the other atoms are at rest. Compute the effective spring constant. One mole of silver has a mass of 108 g and contains 6.023×10^{23} atoms. [**Hints:** $k = \omega^2 m = 4\pi^2\nu^2 m$]

18. Suppose that in Fig. 1.5 the 100 g mass initially moves to the left at a speed of 10 m/s. It strikes the spring and becomes attached to it. (*a*) How far does it compress the spring? (*b*) If the system then oscillates back and forth, what is the amplitude of the oscillation?

$$\left[\textbf{Hints: } \frac{1}{2}(0.1\text{ kg})(10\text{ m/s})^2 = \frac{1}{2}\times(500\text{ N/m})x_0^2\right]$$

19. Suppose that in Fig. 1.5 the 100 g mass compresses the spring 10 cm and is then released. After sliding 50 cm along the flat table from the point of release the mass comes to rest. How large a friction force opposes the motion?

20. A mass of 200 g placed at the lower end of a vertical spring stretches it 20 cm. When it is in equilibrium the mass is hit upward and due to this it goes up a distance of 8 cm before coming down. Find (*a*) the magnitude of the velocity imparted to the mass when it is hit, (*b*) the period of motion.

21. With a 100 g mass at its end a spring executes SHM with a frequency of 1 Hz. How much work is done in stretching the spring 10 cm from its unstretched length?

22. A popgun uses a spring for which *k* = 30 N/cm. When cocked the spring is compressed 2 cm. How high can the gun shoot a 4 g projectile?

23. A block of mass *M*, at rest on a horizontal frictionless table, is attached to a rigid support by a spring of spring constant *k*. A bullet of mass *m* and velocity *v* strikes the block as shown in Fig. 1.38. The bullet remains embedded in the block. Determine

(a) the velocity of the block immediately after the collisions and (b) the amplitude of the resulting simple harmonic motion.

$$\left[\textbf{Hints: } mv = (M+m)V\ ;\ \frac{1}{2}(M+m)V^2 = \frac{1}{2}kA^2\right]$$

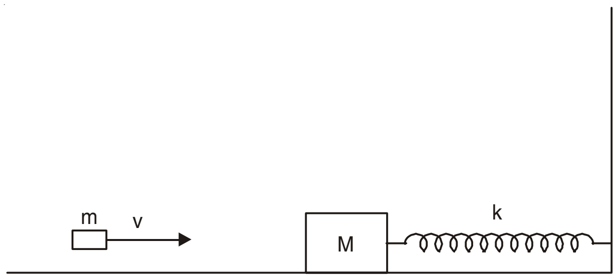

Fig. 1.43

24. A 500 g mass at the end of a Hookean spring vibrates up and down in such a way that it is 2 cm above the table top at its lowest point and 12 cm above the table top at its highest point. Its period is 5s. Find (a) the spring constant, (b) the amplitude of vibration, (c) the speed and acceleration of the mass when it is 10 cm above the table top.

25. A thin metallic wire of length L and area of cross-section A is suspended from free end which stretches it through a distance l. Show that the vertical oscillations of the system are simple harmonic in nature and its time period is given by

$$T = 2\pi\sqrt{l/g} = 2\pi\sqrt{\frac{mL}{(AY)}}$$

where Y is the Young's modulus of the material of the wire.

26. There are two spring systems (a) and (b) of Fig. 1.10 with k_1 = 5 kN/m and k_2 = 10 kN/m. A 100 kg block is suspended from each system. If the block is constrained to move in the vertical direction only, and is displaced 0.01 m down from its equilibrium position, determine for each spring system: (1) The equivalent single spring constant, (2) Time period of vibration, (3) The maximum velocity of the block, and (4) The maximum acceleration of the block.

27. A 10 kg electric motor is mounted on four vertical springs, each having a spring constant of 20 N/cm. Find the frequency with which the motor vibrates vertically.

28. A spring of force constant k is cut into three equal parts, the force constant of each part will be ..... . (*I.I.T. 1978*)

29. A horizontal spring system of mass M executes SHM. When the block is passing through its equilibrium position, an object of mass m is put on it and the two move together. Show that the new frequency and the new amplitude in terms of old frequency and old amplitude are given by

$$\omega' = \omega\sqrt{\frac{M}{(M+m)}},\ A' = A\sqrt{\frac{M}{(M+m)}}.$$

30. Find the period of small oscillations in the vertical plane performed by a ball of mass m = 50 g fixed at the middle of a horizontally stretched string l = 1.0 m in length. The tension of the string is assumed to the constant and equal to T = 10 N.

31. A body of mass m on a horizontal frictionless plane is attached to two horizontal springs of spring constants k_1 and k_2 and equal relaxed lengths L. Now the free ends of the springs are pulled apart and fastened to two fixed walls a distance $3L$ apart. Find the elongations of the springs k_1 and k_2 at the equilibrium position of the body and the time period of small longitudinal oscillations about the equilibrium position.

32. A non-deformed spring whose ends are fixed has a stiffness $k = 12$ N/m. A small body of mass 12 g is attached on the spring at a distance 1/3 l from one end of the spring where l is the length of the spring. Neglecting the mass of the spring find the period of small longitudinal oscillations of the body. Assume that the gravitational force is absent.

[**Hints:** The spring of length $\frac{1}{3}l$ has stiffness $k_1 = \frac{lk}{\frac{1}{3}l} = 3k$ and the spring of length $\frac{2}{3}l$ has stiffness $k_2 = \frac{lk}{\frac{2}{3}l} = \frac{3}{2}k$.]

33. A uniform spring whose unstretched length is L has a force constant k. The spring is cut into two pieces of unstretched lengths L_1 and L_2, with $L_1 = nL_2$. What are the corresponding force constants k_1 and k_2 in terms of n and k?

34. Two bodies of masses m_1 and m_2 are interconnected by a weightless spring of stiffness k and placed on a smooth horizontal surface. The bodies are drawn closer to each other and released simultaneously. Show that the natural oscillation frequency of the system is

$$\omega = \sqrt{k/\mu} \qquad \text{where } \mu = \frac{m_1 m_2}{m_1 + m_2}.$$

35. A particle executes SHM with an amplitude A. At what displacement will the K.E. be equal to twice the P.E.?

36. A body of mass 0.1 kg is connected to three identical springs of spring constant $k = 1$ N/m and in their relaxed state the springs are fixed to three corners of an equilateral triangle ABC (Fig. 1.44). Relaxed length of each spring is 1m. The mass m is displaced from the initial position O to the point D, the mid-point of BC and then released from rest. What will be the kinetic energy of m if it returns to the point O? What will be the speed of the body at O?

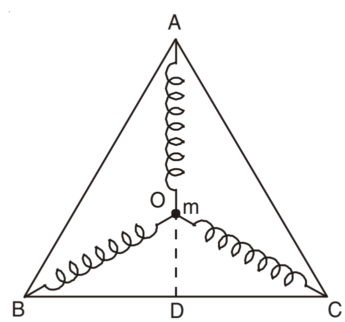

Fig. 1.44

37. Find the length of a second pendulum ($T = 2$ s) at a place where $g = 9.8$ m/s^2.

38. Compare the period of the simple pendulum at the surface of the earth to that at the surface of the moon.

39. The time periods of a simple pendulum on the earth's surface and at a height h from the earth's surface are T and T' respectively. Show that the radius (R) of the earth is given by

$$R = \frac{Th}{T' - T}.$$

40. A simple pendulum of length L and mass (bob) M is oscillating in a plane about a vertical line between angular limits $-\phi$ and $+\phi$. For an angular displacement $\theta\,(|\theta| < \phi)$ the tension in the string and the velocity of the bob are T and v respectively. The following relations hold under the above conditions [Tick the correct relations] :

(*a*) $T \cos \theta = Mg$.

(*b*) $T - Mg \cos \theta = Mv^2/L$

(*c*) The magnitude of the tangential acceleration of the bob $|a_T| = g \sin \theta$.

(*d*) $T = Mg \cos \theta$. (*I.I.T. 1986*)

41. A simple pendulum of length l and mass m is suspended in a car that is travelling with a constant speed v around a circular path of radius R. If the pendulum executes small oscillations about the equilibrium position, what will be its time period of oscillation?

42. A simple pendulum of length l and having a bob of mass m and density ρ is completely immersed in a liquid of density σ ($\rho > \sigma$). Find the time period of small oscillation of the bob in the liquid.

43. Solve problem 27 (Fig. 1.18) by summing the torques about the point O.

44. The mass and diameter of a planet are twice those of the earth. What will be the period of oscillation of a pendulum on this planet if it is a second's pendulum on the earth? (*I.I.T. 1973*)

45. One end of a long metallic wire of length L is tied to the ceiling. The other end is tied to a massless spring of spring constant k. A mass m hangs freely from the free end of the spring. The area of cross-section and the Young's modulus of the wire are A and Y respectively. If the mass is slightly pulled down and released show that it will oscillate with a time period T equal to

$$2\pi\sqrt{\frac{m(YA + kL)}{(YAk)}}. \qquad (\textit{I.I.T. 1993})$$

[**Hints:** If x_1 and x_2 are elongations of metallic wire and spring due to force F, then

$$F = -AYx_1/L = -kx_2$$

and

$$x = x_1 + x_2 = -F\left(\frac{L}{AY} + \frac{1}{k}\right).$$

46. A simple pendulum of mass M is suspended by a thread of length l when a bullet of mass m hits the bob horizontally and sticks in it. The system is deflected by an angle α, where $\alpha < 90°$. Show that the speed of the bullet is

$$\frac{2(M + m)}{m} \sin\left(\frac{\alpha}{2}\right)\sqrt{gl}.$$

47. A cylinder having axis vertical floats in a liquid of density ρ. It is pushed down slightly and released. Find the period of oscillations if the cylinder has weight W and cross-sectional area A.

48. A vertical U-tube of uniform cross-section contains a liquid of total mass M. The mass of the liquid per unit length is m. When disturbed the liquid oscillates back and forth from arm to arm. Calculate the time period if the liquid on one side is depressed and then released. Compute the effective spring constant of the motion.

49. Two identical positive point charge + Q each, are fixed at a distance of $2a$ apart. A point negative charge $(-q)$ of mass m lies midway between the fixed charges. Show that for a small displacement perpendicular to the line joining the fixed charges, the charge $(-q)$ executes SHM and the frequency of oscillations is

$$\frac{1}{2\pi}\sqrt{\frac{Qq}{2\pi\epsilon_0 a^3 m}}$$

50. A thin fixed ring of radius 1 m has a positive charge of 1×10^{-5}C uniformly distributed over it. A particle of mass 0.9 g and having a negative charge 1×10^{-6}C is placed on the axis at a distance of 1 cm from the centre of the ring. Show that the motion of the negative charged particle is approximately simple harmonic. Calculate the time period of oscillations. (*I.I.T. 1982*)

51. A simple pendulum consists of a small sphere of mass m suspended by a thread of length l. The sphere carries a positive charge q. The pendulum is placed in a uniform electric field of strength E directed vertically upwards. With what period the pendulum oscillates if the electrostatic force acting on the sphere is less than the gravitational force?

Assume that the oscillations are small) (*I.I.T. 1977*)

[**Hints:** Net downward force acting on the pendulum is $ma = mg - Eq$]

52. A 2.0 g particle at the end of a spring moves according to the equation

$$y = 0.1 \sin 2\pi t \text{ cm}$$

where t is in seconds. Find the spring constant and the position of the particle at time $t = \frac{1}{\pi}$ s.

53. A particle moves according to the equation

$$y = \frac{1}{\sqrt{2}}\sin 10\sqrt{2}\,t + \frac{1}{10}\cos 10\sqrt{2}\,t.$$

Find the amplitude of the motion.

54. A particle vibrates about the origin of the coordinates along the y-axis with a frequency of 15 Hz and an amplitude of 3.0 cm. The particle is at the origin at time $t = 0$. Find its equation of motion.

55. A particle of mass m moves along the x-axis, attracted toward the origin O by a force proportional to the distance from O. Initially the particle is at distance x_0 from O and is given a velocity of magnitude v_0 (*a*) away from O (*b*) toward O. Find the position at any time, the amplitude and maximum speed in each case.

56. An object of mass 2 kg moves with SHM on the x-axis. Initially ($t = 0$) it is located at a distance 2 m away from the origin $x = 0$, and has velocity 4 m/s and acceleration 8 m/s^2 directed toward $x = 0$. Find (*a*) the position at any time (*b*) the amplitude and period of oscillations, (c) the force on the object when $t = \pi/8$ s.

57. A point particle of mass 0.1 kg is executing SHM of amplitude 0.1 m. When the particle passes through the mean position, its kinetic energy is 8×10^{-3} J. Obtain the equation of motion of the particle if the initial phase of oscillation is 45°.

(*Roorkee 1991*)

58. Retaining terms up to k^2 in problem 49 (page 35) show that the time period of the pendulum is given approximately by

$$T = 2\pi\sqrt{\frac{l}{g}}\left(1+\frac{\psi_0^2}{16}\right)$$

where ψ_0 is the maximum angle made by the string with the vertical.

59. The potential energy of a particle of mass m is given by

$$V(x) = (1 - ax)\exp(-ax), \quad x \geq 0$$

where a is a positive constant. Find the location of the equilibrium point(s), the nature of the equilibrium, and the period of small oscillations that the particle performs about the equilibrium position.

60. An engineer wants to find the moment of inertia of an odd-shaped object about an axis passing through its centre of mass. The object is supported with a wire through its centre of mass along the desired axis. The wire has a torsional constant C = 0.50 Nm. The engineer observes that this torsional pendulum oscillates through 20 complete cycles in 50s. What value of moment of inertia is obtained?

61. A 90 kg solid sphere with a 10 cm radius is suspended by a vertical wire attached to the ceiling of a room. A torque of 0.20 Nm is required to twist the sphere through an angle of 0.85 rad. What is the period of oscillation when the sphere is released from this position?

62. Compare the time periods of vibrations of two loaded light cantilevers made of the same material and having the same length and weight at the free end with the only difference that while one has a circular cross-section of radius a, the other has a square cross-section, each side of which is equal to a.

63. A long horizontal wire AB, which is free to move in a vertical plane and carries a steady current of 20 A, is in equilibrium at a height of 0.01 m over another parallel long wire CD, which is fixed in a horizontal plane and carries a steady current of 30 A, as shown in Fig. 1.45. Show that when AB is slightly depressed, it executes simple harmonic motion. Find the period of oscillations. (*I.I.T. 1994*)

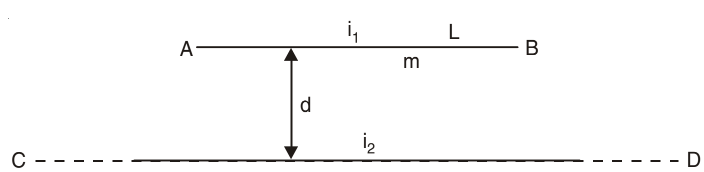

Fig. 1.45

Hints: $\dfrac{\mu_0 i_1 i_2 L}{2\pi d} = mg$. If d is changed to $d - x$, then the restoring force is

$$F = -\frac{\mu_0 i_1 i_2 L}{2\pi(d-x)} + mg \approx -\frac{\mu_0 i_1 i_2 L}{2\pi d^2}x = -\frac{mgx}{d}$$

64. You have a 2.0 mH inductor and wish to make an LC circuit whose resonant frequency can be tuned across the AM radio band (550 kHz to 1600 kHz). What range of capacitance should your variable capacitor cover?

65. An object of mass 0.2 kg executes simple harmonic oscillations along the x-axis with a frequency of $(25/\pi)$ Hz. At the position $x = 0.04$ m, the object has kinetic energy 0.5 J and potential energy 0.4 J. Find the amplitude of oscillations. (*I.I.T. 1994*)

$$\left[\textbf{Hints: } \frac{\text{K.E.}}{\text{P.E.}} = \frac{(A^2 - x^2)}{x^2}\right]$$

66. T_1 is the time period of a simple pendulum. The point of suspension moves vertically upwards according to $y = kt^2$ where $k = 1$ m/s^2. Now the time period is T_2. Then $\dfrac{T_1^2}{T_2^2}$ is ($g = 10$ m/s^2)

(*a*) $\dfrac{4}{5}$ (*b*) $\dfrac{6}{5}$

(*c*) $\dfrac{5}{6}$ (*d*) 1 (*I.I.T. 2005*)

[**Hints :** Upward acceleration of the point of suspension is $a = 2k = 2$ m/s^2 and in this case the effective g is $(10 + 2)$ m/s^2]

67. A simple pendulum has a time period T_1 when on the earth's surface and T_2 when taken to a height R above the earth's surface where R is the radius of the earth. Show that the value of $(T_2/T_1)\dfrac{T_2}{T_1}$ is 2.

$$\left[\textbf{Hints: } mg = G\frac{Mm}{r^2}, T_1 = 2\pi\sqrt{l\Big/\frac{GM}{R^2}}, T_2 = 2\pi\sqrt{l\Big/\frac{GM}{4R^2}}\right]$$

68. A particle executes simple harmonic motion between $x = -A$ to $x = +A$. The time taken for it to go from 0 to $A/2$ is T_1 and to go from $A/2$ to A is T_2. Show that $T_2/T_1 = 2$.

$$\left[\textbf{Hints: } x = A\sin\omega t,\ \omega T_1 = \frac{\pi}{6},\ \omega(T_1 + T_2) = \frac{\pi}{2}\right]$$