Applications of Partial Differential Equations

5.1 INTRODUCTION

The problems related to fluid mechanics, solid mechanics, heat transfer, wave equation and other areas of physics are designed as Initial Boundary Value Problems consisting of partial differential equations and initial conditions.

These problems can be solved by “Method of separation of variables,” in this unit we derive and solve one dimensional heat equation, wave equation, Laplace’s equation in two dimensions etc. by separation of variables method. The general solution of partial differential equation consists arbitrary functions which can be obtained by Fourier Series.

5.2 METHOD OF SEPARATION OF VARIABLES

In this method, we assume that the required solution is the product of two functions i.e.,

\[ u(x, y) = X(x)Y(y) \]  

...(i)

Then we substitute the value of \(u(x, y)\) from (i) and its derivatives reduces the P.D.E. to the form

\[ f_1(X, X', \cdots) = f_2(Y, Y', \cdots) \]  

...(ii)

which is separable in \(X\) and \(Y\). Since \(f_2(Y, Y', \cdots)\) is function \(Y\) only and \(f_1(X, X', \cdots)\) is function of \(X\) only, then equation (ii) must be equal to a common constant say \(k\). Thus (ii) reduces to

\[ f_1(X, X', \cdots) = f_2(X, X', \cdots) = k. \]

**Example 1:** Using, the method of separation of variables solve \(\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u\),

where \(u(x, 0) = 6e^{-3x}\).  

*(U.P.T.U. 2006)*

**Solution:** We have

\[ \frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \]  

...(1)

Let \(u = X(x) \cdot T(t)\)  

...(2)
then
\[ \frac{\partial u}{\partial x} = \frac{\partial X}{\partial x} \frac{dX}{dT} = \frac{dX}{dx} T \]
As \( X \) is the function of \( x \) alone
and
\[ \frac{\partial u}{\partial t} = X \frac{\partial T}{\partial t} = X \frac{dT}{dt} \]

Putting these values in equation (1), we get
\[ T \frac{dX}{dx} = 2X \frac{dT}{dt} + XT \]
\[ \Rightarrow \frac{1}{X} \frac{dX}{dx} = \frac{2}{T} \frac{dT}{dt} + 1 \quad \text{(On dividing by } XT) \]
\[ \Rightarrow \frac{1}{X} \frac{dX}{dx} = \frac{2}{T} \frac{dT}{dt} + 1 = k \]
Now
\[ \frac{1}{X} \frac{dX}{dx} = k \Rightarrow \frac{dX}{X} = k dx \quad \text{...(3)} \]

On integrating, we get
\[ \log e X = kx + \log e c_1 \]
\[ \Rightarrow \log e \frac{X}{c_1} = kx \Rightarrow X = c_0 e^{kx}. \]
and again from (3), we get
\[ \frac{2}{T} \frac{dT}{dt} + 1 = k \]
\[ \Rightarrow \frac{2}{T} \frac{dT}{dt} = k - 1 \Rightarrow \frac{dT}{T} = \frac{k-1}{2} dt \]
On integrating, we obtain
\[ \log e T = \frac{k-1}{2} t + \log e c_2 \Rightarrow \log e \frac{T}{c_2} = \frac{k-1}{2} t \]
\[ \Rightarrow T = c_3 e^{\left(\frac{k-1}{2}\right) t}. \]

Putting the values of \( X \) and \( T \) is equation (2), we get
\[ u = c_1 e^{kx} \cdot c_2 e^{\left(\frac{k-1}{2}\right) t} \]
\[ \Rightarrow u = c_1 c_2 e^{kx + \left(\frac{1}{2}(k-1)\right) t} \quad \text{...(4)} \]
On putting \( t = 0 \) and \( u = 6e^{-3x} \) in (4), we have
\[ 6e^{-3x} = c_1 c_2 e^{kx} \Rightarrow c_1 c_2 = 6 \] and \( k = -3 \)
Hence the required solution of given equation is
\[ u = 6e^{-3x} + \frac{1}{2}(-3-1)t = 6e^{-3x} - 2t \]
\[ \Rightarrow \quad u = 6e^{-3x}e^{-2t} \quad \text{Ans.} \]

**Example 2:** Solve the following equation by the method of separation of variables.

\[ \frac{\partial u}{\partial x} = 4 \frac{\partial u}{\partial y} \quad \text{where} \quad u(0,y) = 8e^{-3y} \quad \text{(U.P.T.U. 2008)} \]

**Solution:** We have

\[ \frac{\partial u}{\partial x} = 4 \frac{\partial u}{\partial y} \quad \cdots (1) \]

Let

\[ u = Xy \quad \cdots (2) \]

then

\[ \frac{\partial u}{\partial x} = \frac{dX}{dx} \frac{\partial u}{\partial x} + \frac{dY}{dy} \frac{\partial u}{\partial y} \]

Putting these values in equation 1, we get

\[ Y \frac{dx}{dX} = 4X \frac{dY}{dy} \]

or

\[ \frac{1}{X} \frac{dX}{dx} = \frac{4}{Y} \frac{dY}{dy} = k \]

Now

\[ \frac{1}{X} \frac{dX}{dx} = K \Rightarrow \frac{dX}{X} = kdx \Rightarrow \log_e X = kx + \log_e c_1 \]

or

\[ X = c_1 e^{kx} \]

And

\[ \frac{4}{Y} \frac{dY}{dy} = k \Rightarrow \frac{dY}{Y} = \frac{k}{4}dy \Rightarrow \log_e Y = \frac{k}{4}y + \log_e c_2 \]

or

\[ Y = c_2 e^{\frac{ky}{4}} \]

\[ \therefore \quad \text{From (2), we get} \]

\[ u = c_1 c_2 e^{\left(4x + \frac{ky}{4}\right)} \quad \cdots (3) \]

Putting \( x = 0 \), in the equation (3), we get

\[ 8e^{-3y} = c_1 c_2 e^{\frac{ky}{4}} \]
Comparing on both sides
\[ c_1 c_2 = 8 \quad \text{and} \quad \frac{k}{4} = -3 \quad \Rightarrow \quad k = -12 \]
Hence
\[ u = 8e^{-12x+3y} \quad \text{Ans.} \]

Example 3: Solve by the method of separation of variables
\[ \frac{\partial^2 z}{\partial x^2} - 2 \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0. \quad (U.P.T.U. 2005) \]

Solution: We have
\[ \frac{\partial^2 z}{\partial x^2} - 2 \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0 \] ... (1)
Let \[ z = X(x)Y(y) \] ... (2)
\[ \Rightarrow \frac{\partial z}{\partial x} = \frac{dX}{dx} Y \Rightarrow \frac{\partial^2 z}{\partial x^2} = \frac{d^2 X}{dx^2} Y \]
and
\[ \frac{\partial z}{\partial y} = \frac{dY}{dy} X \]
Putting these values in equation (1), we get
\[ Y \frac{d^2 X}{dx^2} - 2Y \frac{dX}{dx} + X \frac{dY}{dy} = 0 \]
\[ \Rightarrow \frac{1}{X} \frac{d^2 X}{dx^2} - \frac{2}{X} \frac{dX}{dx} + \frac{1}{Y} \frac{dY}{dy} = 0 \] (divide by XY)
\[ \Rightarrow \frac{1}{X} \frac{d^2 X}{dx^2} - \frac{2}{X} \frac{dX}{dx} = \frac{1}{Y} \frac{dY}{dy} = k \] ... (3)
\[ \text{Now} \quad \frac{1}{X} \frac{d^2 X}{dx^2} - \frac{2}{X} \frac{dX}{dx} = k \quad \Rightarrow \quad (D^2 - 2D - k)X = 0 \]
The A.E. is \( m^2 - 2m - k = 0 \)
\[ \Rightarrow \quad m = \frac{2 \pm \sqrt{4 + 4k}}{2} \quad \Rightarrow \quad m = 1 \pm \sqrt{1 + k} \]
\[ \therefore \quad X = c_1 e^{\{1+\sqrt{1+k}\}x} + c_2 e^{\{1-\sqrt{1+k}\}x} \]
and again from (3), we have
\[
\frac{1}{Y} \frac{dY}{dy} = -k \Rightarrow \frac{dY}{Y} = -kdy
\]
On integrating, we get
\[
\log_e Y = -ky + \log c_3
\]
\[
\Rightarrow \quad \log_e \frac{Y}{c_3} = -ky \Rightarrow Y = c_3 e^{-ky}
\]
Hence the required solution of equation (1) is
\[
z = \left[ c_1 e^{k x (1 + \sqrt{1-k})} + c_2 e^{k x (1 - \sqrt{1-k})} \right] c_3 e^{-ky}.
\]
**Ans.**

**Example 4:** Solve by the method of separation of variables \( \frac{\partial u}{\partial x} = 2 \frac{\partial u}{\partial y} + u \), where \( u(x,0) = 3e^{-5x} - 2e^{-3x} \).

**Solution:** We have
\[
\frac{\partial u}{\partial x} = 2 \frac{\partial u}{\partial y} + u \quad \ldots(1)
\]
Let \( u = XY \) \ldots(2)

Putting this value in equation (1), we get
\[
\frac{\partial}{\partial x} (XY) = 2 \frac{\partial}{\partial y} (XY) + XY
\]
\[
\Rightarrow \quad Y \frac{dX}{dx} = 2X \frac{dY}{dy} + XY
\]
\[
\Rightarrow \quad \frac{1}{X} \frac{dX}{dx} = \frac{2}{Y} \frac{dY}{dy} + 1 = k \quad \ldots(3)
\]
Now
\[
\frac{1}{X} \frac{dX}{dx} = k \Rightarrow \frac{dX}{X} = kdx
\]
On integrating, we get
\[
\log_e X = kx + \log_e c_1
\]
\[
\Rightarrow \quad X = c_1 e^{kx}.
\]
And taking last two terms of equation (3), we have
\[
\frac{2}{Y} \frac{dY}{dy} + 1 = k \Rightarrow \frac{2}{Y} \frac{dY}{dy} = k - 1
\]
On integrating, \( \log_e Y = \frac{(k - 1)}{2} y + \log_e c_2 \)

\[ Y = c_2 e^{(k-1)y/2}. \]

From (2), we get

\[ u = c_1 c_2 e^{k_1 t} \cdot e^{(k-1)y/2} \]

\[ u = \sum_{n=1}^{\infty} b_n e^{k_n x} e^{(k-1)y/2} \]

\[ (b_n = c_1 c_2) \text{ and } (k = k_n) \]

which is the most general solution of equation (1).

Putting \( y = 0 \) and \( u = 3e^{-5x} - 2e^{-3x} \) in equation (4), we get

\[ 3e^{-5x} - 2e^{-3x} = \sum_{n=1}^{\infty} b_n e^{k_n x} = b_1 e^{k_1 x} + b_2 e^{k_2 x} \]

Comparing the terms on both sides, we get

\[ b_1 = 3, \quad k_1 = -5, \quad b_2 = -2, \quad k_2 = -3 \]

Hence the required solution of given equation is from (4), we have

\[ u = 3e^{-5x} \cdot e^{-3y} + (-2)e^{-3x} \cdot e^{-2y} \]

\[ u = 3e^{-(5x + 3y)} - 2e^{-(3x + 2y)}. \quad \text{Ans.} \]

### 5.3 ONE DIMENSIONAL WAVE EQUATION

(U.P.T.U. 2007)

The one dimensional wave equation arises in the study of transverse vibrations of an elastic string. Consider an elastic string, stretched to its length \( l \) between two points \( O \) and \( A \) fixed. Let the function \( y(x, t) \) denote the displacement of string at any point \( x \) and at any time \( t > 0 \) from the equilibrium position (\( x \)-axis). When the string released after stretching then it vibrates and therefore the transverse vibrations formed a one dimensional wave equation.

Let the string is perfectly flexible and does not offer resistance to bending. Let \( T_1 \) and \( T_2 \) be tensions at the end points \( P \) and \( O \) of the portion of the string. Since there is no motion in the horizontal direction. Thus the sum of the forces in the horizontal direction must be zero \( i.e., \)

\[ -T_1 \cos \alpha + T_2 \cos \beta = 0 \]

\[ T_1 \cos \alpha = T_2 \cos \beta = T \] (constant) \[ \quad \text{...(i)} \]
Let \( m \) be the mass of the string per unit length then the mass of portion \( PQ = m \delta s \).

Now by Newton’s second law of motion, the equation of motion in the vertical direction is

\[
\text{mass} \times \text{acceleration} = \text{resultant of forces}
\]

\[
m \delta s \frac{d^2y}{dt^2} = T_2 \sin \beta - T_1 \sin \alpha \tag{\( ii \)}
\]

Dividing \( (ii) \) by \( (i) \), we have

\[
\frac{m \delta x}{T} \frac{d^2y}{dt^2} = \frac{T_2 \sin \beta}{T_2 \cos \beta} - \frac{T_1 \sin \alpha}{T_1 \cos \alpha}
\]

\[
\Rightarrow \quad \frac{m \delta x}{T} \frac{d^2y}{dt^2} = \tan \beta - \tan \alpha
\]

\[
\Rightarrow \quad \frac{d^2y}{dt^2} = \frac{T}{m \delta s} (\tan \beta - \tan \alpha)
\]

\[
= \frac{T}{m} \left( \frac{\frac{dy}{dx}}{\delta x + \delta x} - \frac{\frac{dy}{dx}}{\delta x} \right) \left\mid \delta x \to \delta x \right.
\]

Taking \( \delta x \to 0 \)

\[
\Rightarrow \quad \frac{d^2y}{dt^2} = \frac{T}{m} \lim_{\delta x \to 0} \left\{ \frac{\frac{dy}{dx}}{\delta x} - \frac{\frac{dy}{dx}}{\delta x} \right\}
\]
\[ \Rightarrow \quad \frac{\partial^2 y}{\partial t^2} = \frac{T}{m} \frac{\partial^2 y}{\partial x^2} \]

Thus

\[ \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}. \] ...

... (iii)

where \( c^2 = \frac{T}{m} \). Equation (iii) is known as one dimensional wave equation.

### 5.3.1 Solution of One Dimensional Wave Equation

The wave equation is

\[ \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \] ...

... (i)

Let

\[ y = X(x) \quad T(t). \] ...

where \( X \) is the function of \( x \) only and \( T \) is the function of \( t \) only.

Then

\[ \frac{\partial^2 y}{\partial t^2} = X \frac{d^2 T}{dt^2} \quad \text{and} \quad \frac{\partial^2 y}{\partial x^2} = T \frac{d^2 X}{dx^2} \]

Putting these values in equation (i), we get

\[ X \frac{d^2 T}{dt^2} = c^2 T \frac{d^2 X}{dx^2} \]

\[ \Rightarrow \quad \frac{1}{X} \frac{d^2 X}{dx^2} = \frac{1}{c^2 T} \frac{d^2 T}{dt^2} = k \]

... (iii)

Now

\[ \frac{1}{X} \frac{d^2 X}{dx^2} = k \]

\[ \Rightarrow \quad (D^2 - k)X = 0; \quad \frac{d}{dx} = D \]

The A.E. is \( m^2 - k = 0 \)

\[ m = \pm \sqrt{k} \]

\[ \therefore \quad X = c_1 e^{\sqrt{k}x} + c_2 e^{-\sqrt{k}x} \]

and, again from (iii), we get

\[ \frac{d^2 T}{dt^2} = kc^2 T \quad \Rightarrow \quad (D^2 - kc^2)T = 0; \quad D = \frac{d}{dt} \]

\[ \Rightarrow \quad \text{The A.E. is} \quad m^2 - kc^2 = 0 \quad \Rightarrow \quad m = \pm c\sqrt{k} \]
Thus, from equation (ii), we get
\[ y = (c_1 e^{\sqrt{k^2} x} + c_2 e^{-\sqrt{k^2} x})(c_3 e^{\sqrt{k^2} t} + c_4 e^{-\sqrt{k^2} t}) \]

There are arise following cases:

**Case I:** If \( k > 0 \) let \( k = p^2 \)
then
\[ y = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{cpe} + c_4 e^{-cpe}) \] \( \ldots (A) \)

**Case II:** If \( k < 0 \), let \( k = -p^2 \)
then
\[ m^2 = -p^2 \Rightarrow m = \pm pi \]
\[ X = c_1 \cos px + c_2 \sin px \]
and
\[ m^2 = -p^2 c^2 \Rightarrow m = \pm ipc \]
\[ T = c_3 \cos cpe + c_4 \sin cpe \]
then
\[ y = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpe + c_4 \sin cpe) \] \( \ldots (B) \)

**Case III:** If \( k = 0 \)
then
\[ D^2 X = 0 \Rightarrow m = 0, 0 \]
\[ \therefore X = (c_1 + c_2 x) \]
And
\[ D^2 T = 0 \Rightarrow m = 0, 0 \]
\[ \therefore T = (c_3 + c_4 t) \]
Then
\[ y = (c_1 + c_2 x)(c_3 + c_4 t). \] \( \ldots (C) \)

Of these three solutions, we have choose the solution which is consistent with the physical nature of the problem.

Since the physical nature of one dimension wave equation is periodic so we consider the solution which has periodic nature.

Here only the solution in equation \((B)\) is periodic (as both sine and cosine are periodic).

Thus, the desired solution, for one dimensional wave equation is
\[ y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpe + c_4 \sin cpe). \] \( \ldots (iv) \)

Now using the boundary conditions
At \( x = 0 \) (origin), \( y = 0 \) from \((iv)\), we get
\[ 0 = c_1(c_3 \cos cpe + c_4 \sin cpe) \Rightarrow c_1 = 0 \]
using the value of \( c_1 \) in \((iv)\), we get
\[ y(x, t) = c_2 \sin px(c_3 \cos cpe + c_4 \sin cpe) \] \( \ldots (v) \)

At \( x = l \) (at A), \( y = 0 \) from \((v)\), we get
\[ 0 = c_2 \sin pl(c_3 \cos cpe + c_4 \sin cpe) \]
APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

\[ \sin \frac{nl}{L} = 0 = \sin \frac{n\pi}{l} \quad \Rightarrow \quad pl = n\pi \]

\[ p = \frac{n\pi}{l}, \quad \text{where } n = 1, 2, 3, \ldots \]

Hence, the solution of wave equation satisfying the boundary conditions is, from (v)

\[ y(x, t) = c_2 \sin \frac{n\pi x}{L} \left( c_3 \cos \frac{n\pi ct}{l} + c_4 \sin \frac{n\pi ct}{l} \right) \quad \bigg| \quad p = \frac{n\pi}{l} \]

\[ = \left( a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l} \quad \bigg| \quad c_2c_3 = a_n \]

\[ c_2c_4 = b_n \]

\[ \therefore \quad \text{The general solution of wave equation is} \]

\[ y(x, t) = \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l} \quad \ldots (vi) \]

**Remark:** We can apply initial conditions on above equation (vi) in time domain i.e., at \( t = 0 \).

**Example 5:** A string of length \( l \) is fastened at both ends \( A \) and \( C \). At a distance ‘\( b \)’ from the end \( A \), the string is transversely displaced to a distance ‘\( d \)’ and is released from rest when it is in this position. Find the equation of the subsequent motion.

**Solution:** Let \( y(x, t) \) is the displacement of the string.

Now, by the one dimensional wave equation, we have

\[ \frac{\partial^2 y}{\partial t^2} = c_2 \frac{\partial^2 y}{\partial x^2} \quad \ldots (1) \]

The solution of equation (1) is (from equation (iv), page 444)
Now using the boundary conditions as follows:

The boundary conditions are

(i) At $x = 0$ (at $A$), \( y = 0 \) \( \Rightarrow \) \( y(0, t) = 0 \)

and

(ii) At $x = l$ (at $C$), \( y = 0 \) \( \Rightarrow \) \( y(l, t) = 0 \)

From (2), we have

\[
0 = c_1(c_3 \cos cpt + c_4 \sin cpt) \quad \Rightarrow \quad c_1 = 0
\]

Using \( c_1 = 0 \), in equation (2), we get

\[
y(x, t) = c_2 \sin px(c_3 \cos cpt + c_4 \sin cpt)
\]

...(3)

Using (ii) boundary condition, from (3), we have

\[
0 = c_2 \sin pl(c_3 \cos cpt + c_4 \sin cpt)
\]

\[
\Rightarrow \quad \sin pl = 0 \quad \Rightarrow \quad \sin pl = \sin n\pi \quad \Rightarrow \quad p = \frac{n\pi}{l}.
\]

Using the value of \( p \) in (3), we obtain

\[
y(x, t) = c_2 \sin \frac{n\pi x}{l} \left( c_3 \cos \frac{n\pi ct}{l} + c_4 \sin \frac{n\pi ct}{l} \right)
\]

...(4)

Next, the initial conditions are as follows:

(iii) \( \frac{\partial y}{\partial t} = 0 \) at \( t = 0 \) and displacement at \( t = 0 \) is

\[
y(x, 0) = \begin{cases} \frac{d \cdot x}{b}, & 0 \leq x \leq b \\ \frac{d(x-l)}{(b-l)}, & b \leq x \leq l \end{cases}
\]

From (4),

\[
\frac{\partial y}{\partial t} = c_2 \sin \frac{n\pi x}{l} \left( -n\pi c_3 \sin \frac{n\pi ct}{l} + n\pi c_4 \cos \frac{n\pi ct}{l} \right)
\]

Using (iii) in above equation, we get

\[
0 = c_2 c_4 \frac{n\pi c}{l} \cdot \sin \frac{n\pi x}{l} \quad \Rightarrow \quad c_4 = 0
\]

Using \( c_4 = 0 \) in equation (4), we get

\[
y(x, t) = c_2 c_3 \sin \frac{n\pi x}{l} \cdot \cos \frac{n\pi ct}{l}
\]

\[.\text{ The general solution of given problem is}\]

\[
y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}
\]

...(5)
Using initial condition (iv) in equation (5), we get

\[ y(x, 0) = \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi x}{l} \right) \]

which is half range Fourier sine series, so we have

\[ b_n = \frac{2}{l} \int_{0}^{l} y(x, 0) \cdot \sin \left( \frac{n\pi x}{l} \right) dx \]

\[ = \frac{2}{l} \int_{0}^{b} \frac{d}{b} x \cdot \sin \left( \frac{n\pi x}{l} \right) dx + \frac{2}{l} \int_{b}^{l} \frac{d}{l(b-l)} (x-l) \sin \left( \frac{n\pi x}{l} \right) dx \]

\[ = \frac{2d}{bl} \left[ \frac{-l}{n\pi} \cos \left( \frac{n\pi x}{l} \right) - \left( \frac{-l^2}{n^2 \pi^2} \right) \sin \left( \frac{n\pi x}{l} \right) \right]_{0}^{b} \]

\[ + \frac{2d}{l(b-l)} \left[ (x-l) \left( \frac{-l}{n\pi} \cos \left( \frac{n\pi x}{l} \right) - \left( \frac{-l^2}{n^2 \pi^2} \right) \sin \left( \frac{n\pi x}{l} \right) \right) \right]_{0}^{l} \]

\[ \Rightarrow b_n = -\frac{2d}{n\pi} \cos \frac{n\pi b}{l} + \frac{2d^2}{b l n^2 \pi^2} \sin \frac{n\pi b}{l} + \frac{2d}{n\pi} \cos \frac{n\pi b}{l} \]

\[ - \frac{2d^2}{l(b-l) n^2 \pi^2} \sin \frac{n\pi b}{l} \]

\[ \Rightarrow b_n = \frac{2d^2}{b(l-b) n^2 \pi^2} \sin \frac{n\pi b}{l} \]

\[ \therefore \text{From (5), we get} \]

\[ y(x, t) = \frac{2d^2}{b(l-b) n^2 \pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi x}{l} \times \sin \frac{n\pi b}{l} \times \cos \frac{n\pi ct}{l}. \quad \text{Ans.} \]

**Example 6:** A string is stretched and fastened to two points \( l \) apart. Motion is started by displacing the string in the form \( y = a \sin \left( \frac{\pi x}{l} \right) \) from which it is released at a time \( t = 0 \). Show that the displacement of any point at a distance \( x \) from one end at time \( t \) is given by

\[ y(x, t) = a \sin \left( \frac{\pi x}{l} \right) \cos \left( \frac{\pi ct}{l} \right) \]

\( (U.P.T.U. \ 2004) \)

**Solution:** Let \( y(x, t) \) be the displacement at any point \( P(x, y) \) at any time.
Then by the wave equation, we have

\[
\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}
\]

...(1)

The solution of equation (1) is of the form

\[
y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)
\]

...(2)

Now using the boundary conditions

(i) At \( x = 0 \), the displacement \( y = 0 \) \( \implies y(0, t) = 0 \)

(ii) At \( x = l \), the displacement \( y = 0 \) \( \implies y(l, t) = 0 \)

Using (i) boundary condition in (2), we get

\[
y(0, t) = 0 = c_1 (c_3 \cos cpt + c_4 \sin cpt) \implies c_1 = 0.
\]

\[
\therefore \text{ From (2), we get } y(x, t) = c_2 \sin px(c_3 \cos cpt + c_4 \sin cpt)
\]

Using (ii) boundary condition in equation (3), we get

\[
y(l, t) = 0 = c_2 \sin pl(c_3 \cos cpt + c_4 \sin cpt)
\]

\[
\implies \sin pl = 0 = \sin n\pi \implies p = \frac{n\pi}{l}.
\]

Now using the initial conditions

(iii) At \( t = 0 \), the velocity \( \frac{\partial y}{\partial t} = 0 \) \( \implies \left( \frac{\partial y}{\partial t} \right)_{t=0} = 0 \)

(iv) At \( t = 0 \), the displacement \( y = a \sin \frac{\pi x}{l} \) \( \implies y(x, 0) = a \sin \frac{\pi x}{l} \)

\[
\therefore \text{ From (3), we have } \frac{\partial y}{\partial t} = c_2 \sin px[c_3(-cp) \sin cpt + c_4(\cos(cpt)]
\]

Using (iii) initial condition in above equation, we get

\[
0 = c_2 c_4 cp \sin px \implies c_2 c_4 cp = 0
\]

\[
\implies c_4 = 0. \quad | c_2 \neq 0, \text{otherwise there is trivial solution}
\]

Using \( p = \frac{n\pi}{l} \) and \( c_4 = 0 \), in equation (3), we get

\[
y(x, t) = c_2 c_3 \sin \frac{\pi x}{l} \cos \frac{\pi c t}{l}
\]
The general solution is

\[ y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \quad (b_n = c_2 c_3) \]  

Finally using (iv) initial condition in equation (4), we get

\[ y(x, 0) = a \sin \frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \]

or

\[ a \sin \frac{\pi x}{l} = b_1 \sin \frac{\pi x}{l} + b_2 \sin \frac{2\pi x}{l} + \ldots \]

Equating the coefficient of \( \sin \frac{\pi x}{l} \), we get

\[ b_1 = a, \quad b_2 = b_3 = \ldots = 0 \]

Hence the required solution of given problem is

\[ y(x, t) = a \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l}; \quad n = 1. \quad \text{Proved.} \]

**Example 7:** Find the displacement of a string stretched between two fixed points at a distance 2\( l \) apart when the string is initially at rest in equilibrium position and points of the string are given initial velocity \( v \) where

\[ v = \begin{cases} \frac{x}{l}, & \text{when } 0 < x < l \\ \frac{2l-x}{l}, & \text{when } l < x < 2l \end{cases} \]

\( x \) being the distance measured from one end.

**Solution:** The displacement \( y(x, t) \) is given by wave equation

\[ \frac{\partial^2 y}{\partial t^2} = c_x^2 \frac{\partial^2 y}{\partial x^2} \]  

The solution of equation is given by

\[ y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin c pt) \]  

Now, the boundary conditions are

(i) At \( x = 0, \quad y = 0 \Rightarrow y(0, t) = 0 \)

(ii) At \( x = 2l, \quad y = 0 \Rightarrow y(2l, t) = 0 \)

Using (i) boundary condition in (2), we get

\[ 0 = c_1(c_3 \cos c pt + c_4 \sin c pt) \quad \Rightarrow \quad c_1 = 0 \]

\[ \therefore \text{From (2), we get} \]

\[ y(x, t) = c_2 \sin px(c_3 \cos c pt + c_4 \sin c pt) \]  

Using \((ii)\) condition in \((3)\), we get

\[
0 = c_2 \sin 2pl(c_3 \cos c + c_4 \sin c)
\]

\[\Rightarrow \quad \sin 2pl = 0 = \sin n\pi \quad \Rightarrow \quad p = \frac{n\pi}{2l}.
\]

Now, the initial conditions are:

\((iii)\) At \(t = 0\) the displacement \(y(x, 0) = 0\).

\((iv)\) At \(t = 0\), \(\frac{\partial y}{\partial t} = v\).

Making use of initial condition \((iii)\) in \((3)\), we get

\[
y(x, 0) = 0 = c_2 \sin px(c_3) \quad \Rightarrow \quad c_3 = 0
\]

\[
\therefore \quad \text{From } (3), \quad y(x, t) = c_2 c_4 \sin \frac{n\pi x}{2l} \sin \frac{n\pi t}{2l} \quad \bigg| \quad p = \frac{n\pi}{2l}
\]

The general solution is

\[
y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2l} \sin \frac{n\pi t}{2l}
\]

\[\Rightarrow \quad \frac{\partial y}{\partial t} = \frac{\pi c}{2l} \sum_{n=1}^{\infty} nb_n \sin \frac{n\pi x}{2l} \cos \frac{n\pi t}{2l}\]

Using initial condition \((iv)\) in above equation, we get

\[
v = \frac{\pi c}{2l} \sum_{n=1}^{\infty} nb_n \sin \frac{n\pi x}{2l}
\]

which represents half range Fourier sine series

\[
\therefore \quad \frac{\pi c}{2l} \int_{-l}^{0} v \sin \frac{n\pi x}{2l} \, dx
\]

\[
= \frac{1}{l} \int_{-l}^{0} x \sin \frac{n\pi x}{2l} \, dx + \frac{1}{l} \int_{0}^{l} \left(2l - x\right) \sin \frac{n\pi x}{2l} \, dx
\]

\[
= \frac{1}{l} \left[ \frac{x}{l} \left(-1\right)^{\frac{2l}{n}} \cos \frac{n\pi x}{2l} - \frac{1}{l} \left(-1\right)^{\frac{4l^2}{n^2\pi^2}} \sin \frac{n\pi x}{2l} \right]_{0}^{l}
\]

\[
+ \frac{1}{l} \left[ \left(-1\right)^{\frac{2l}{n}} \left(-1\right)^{\frac{4l^2}{n^2\pi^2}} \sin \frac{n\pi x}{2l} \right]_{0}^{l}
\]
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\[ y(x, t) = \sum_{n=1}^{\infty} \frac{2}{n\pi c} \sin \left( \frac{n\pi}{2l} \right) \sin \left( \frac{n\pi x}{2l} \right) \cos \left( \frac{n\pi ct}{2l} \right). \]

Hence the displacement function is given by, from equation (4), we get

**Example 8:** A string is stretched and fastened to two points \( l \) apart. Motion is started by displacing the string into the form \( y = k(lx - x^2) \) from which it is released at time \( t = 0 \). Find the displacement of any point on the string at a distance of \( x \) from one end at time \( t \). (U.P.T.U. 2002)

**Solution:** Let the displacement \( y(x, t) \) given by the wave equation

\[ \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \] 

\( \therefore \) \[ y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos c \pi t + c_4 \sin c \pi t) \] 

Using the boundary conditions \( (i) y(0, t) = 0 \) \( (ii) y(l, t) = 0 \)

Using \( (i) \) in equation (2), we get

\[ y = c_2 \sin px (c_3 \cos c \pi t + c_4 \sin c \pi t) \] 

Using \( (ii) \), in equation (3), we get

\[ 0 = c_2 \sin pl (c_3 \cos c \pi t + c_4 \sin c \pi t) \Rightarrow \sin pl = 0 \]

\[ \Rightarrow \sin pl = \sin n\pi \Rightarrow p = \frac{n\pi}{l}. \]

and the initial conditions are

\( (iii) \) \( \left( \frac{\partial y}{\partial t} \right)_{t=0} = 0 \) \( (iv) y(x, 0) = k(lx - x^2) \)

\( \therefore \) From (3), we get

\[ \frac{\partial y}{\partial t} = c_2 \sin px [-c_3 c \sin c \pi t + c_4 c \cos c \pi t] \]

Using \( (iii) \) in above relation, we get

\[ 0 = c_2 (c_4 c) \sin px \Rightarrow c_4 = 0. \]
Using the values of $c$ and $p$ in equation (3), we get

$$y = c_2c_3 \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}$$

$\therefore$ The general solution is

$$y = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \quad \text{...(4)}$$

Making use of (iv) in (4), we get

$$k(lx - x^2) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

which represents half range Fourier sine series

$\therefore$

$$b_n = \frac{2k}{l} \int_0^l (lx - x^2) \sin \frac{n\pi x}{l} \, dx$$

$\Rightarrow$

$$b_n = \frac{2k}{l} \left[ (lx - x^2) \left( -\cos \frac{n\pi x}{l} \right) \frac{l}{n\pi} - (l - 2x) \left( -\sin \frac{n\pi x}{l} \right) \frac{l^2}{n^2 \pi^2} \right]_0^l$$

$$+ (-2) \left( \cos \frac{n\pi x}{l} \right) \frac{l^3}{n^3 \pi^3} \right]_0^l$$

$\Rightarrow$

$$b_n = \frac{2k}{l} \left[ (-1)^{n+1} \frac{2l^3}{n^3 \pi^3} + \frac{2l^3}{n^3 \pi^3} \right] = \frac{8kl^2}{n^3 \pi^3} \text{ when } n \text{ is odd.}$$

$$= 0 \quad \text{when } n \text{ is even.}$$

Hence the required displacement is, from (4), we get

$$y = \sum_{n=1}^{\infty} \frac{8kl^2}{n^3 \pi^3} \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}, \quad \text{when } n \text{ is odd.} \quad \text{Ans.}$$

**Example 9:** A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in its equilibrium position. If it is set vibrating by giving to each of its points on initial velocity $\lambda x(l - x)$, find the displacement of the string at any distance $x$ from one end at any time $t$.


**Solution:** We know that the solution of one dimensional wave equation with boundary conditions

$$y(0, t) = y(l, t) = 0 \quad \text{is}$$

$$y(x, t) = c_2 \sin px(c_3 \cos c t + c_4 \sin c t) \quad \text{...(1)}$$

where $p = \frac{n\pi}{l}$.
Now the initial conditions are

(a) \( y(x,0) = 0 \)  
(b) \( \left( \frac{\partial y}{\partial t} \right)_{t=0} = \lambda x(l-x) \)

Making use of (a) in (1), we get

\[
0 = c_3c_2 \sin px \implies c_3 = 0.
\]

\[\therefore\] From (1), we get

\[
y(x, t) = c_2c_4 \sin \frac{n \pi x}{l} \sin \frac{n \pi ct}{l}
\]

The general solution of wave equation is

\[
y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n \pi x}{l} \sin \frac{n \pi ct}{l} \quad \ldots(2)
\]

From (2),

\[
\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} \left( \frac{n \pi ct}{l} \right) b_n \sin \frac{n \pi x}{l} \cos \frac{n \pi ct}{l}
\]

\[\therefore\]

\[
\left( \frac{\partial y}{\partial t} \right)_{t=0} = \lambda x(l-x) = \sum_{n=1}^{\infty} \left( \frac{n \pi c}{l} \right) b_n \sin \frac{n \pi x}{l}
\]

\[
\frac{n \pi c}{l} b_n = \frac{2}{l} \int_{0}^{l} \lambda x(l-x) \sin \frac{n \pi x}{l} dx
\]

\[
= \frac{2 \lambda}{l} \int_{0}^{l} x(l-x) \left( \frac{l}{n \pi} \cos \frac{n \pi x}{l} \right) d(x(l-x)) - (l-2x)
\]

\[
= \frac{4 \lambda l^2}{n^2 \pi^2} \sin \frac{n \pi x}{l} \right) (+(-2) \left( \frac{l^3}{n^3 \pi^3} \cos \frac{n \pi x}{l} \right)
\]

\[
= \frac{4 \lambda l^2}{n^2 \pi^3} (1- \cos n \pi) = \frac{4 \lambda l^2}{n^2 \pi^3} \left[ (-1)^n \right]
\]

\[\therefore\]

\[
b_n = \begin{cases} \frac{8 \lambda l^3}{cn^2 \pi^4}, & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even} \end{cases}
\]

\[\therefore\] From (2) the required solution is

\[
y(x, t) = \frac{8 \lambda l^3}{c \pi^2} \sum_{n=1}^{\infty} \frac{1}{n^4} \sin \frac{n \pi x}{l} \sin \frac{n \pi ct}{l}, \quad n \text{ is odd. } \text{Ans.}
\]
Example 10: If a string of length \( l \) is initially at rest in equilibrium position and each of its points is given the velocity \( \frac{30}{t} \), find the displacement \( y(x, t) \). 

\( \text{\textit{U.P.T.U. 2001, 2006}} \)

Solution: The equation for vibrations of the string is

\[
\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}
\]

The solution of above equation is

\[
y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt) \quad \text{...(1)}
\]

At \( x = 0 \), the displacement \( y = 0 \)

\[
0 = c_1(c_3 \cos cpt + c_4 \sin cpt) \quad \Rightarrow \quad c_1 = 0.
\]

From (1), we get

\[
y(x, t) = c_2 \sin px(c_3 \cos cpt + c_4 \sin cpt) \quad \text{...(2)}
\]

Again at \( x = l \), the displacement \( y = 0 \)

\[
0 = c_2 \sin pl(c_3 \cos cpt + c_4 \sin cpt)
\]

\[
\Rightarrow \quad \sin pl = 0 = \sin n\pi \quad \Rightarrow \quad p = \frac{n\pi}{l}.
\]

Next making the use of initial conditions \( i.e., \) at \( t = 0 \), the displacement \( y(x, 0) = 0 \).

Again from (2), we get

\[
0 = c_3 \sin px \quad \Rightarrow \quad c_3 = 0
\]

Using \( c_3 = 0 \) and \( p = \frac{n\pi}{l} \), equation (2) takes the form

\[
y(x, t) = c_2c_4 \sin \frac{n\pi x}{l} \sin \frac{n\pi ct}{l}
\]

\[
\Rightarrow \quad \frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} b_n \frac{n\pi x}{l} \sin \frac{n\pi ct}{l}
\]

At \( t = 0 \), \( \left( \frac{\partial y}{\partial t} \right)_{t=0} = b \sin^{\frac{\pi x}{l}} \), we get
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\[ b \sin^3 \frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \frac{n \pi c}{l} \sin \frac{n \pi x}{l} \]

\[ = b_1 \frac{\pi c}{l} \sin \frac{\pi x}{l} + b_2 \frac{2 \pi c}{l} \sin \frac{2 \pi x}{l} + b_3 \frac{3 \pi c}{l} \sin \frac{3 \pi x}{l} + \cdots \]

\[ b \sin^3 \frac{\pi x}{l} = (b_1 + 9b_3) \frac{\pi c}{l} \sin \frac{\pi x}{l} + b_2 \frac{2 \pi c}{l} \sin \frac{2 \pi x}{l} - \frac{12 \pi c}{l} b_3 \sin^3 \frac{\pi x}{l} + \cdots \]

Equating the coefficient of like powers of \( \sin \frac{\pi x}{l} \), we get

\[ b_1 + 9b_3 = 0 \] …(i)

and

\[ \frac{-12 \pi c}{l} b_3 = b \Rightarrow b_3 = -\frac{bl}{12 \pi c} \]

from (i),

\[ b_1 = \frac{3bl}{4 \pi c}, \quad b_2 = b_4 = b_5 = \cdots = 0 \]

Hence from (3), the required displacement is

\[ y(x, t) = \frac{3bl}{4 \pi c} \sin \frac{\pi x}{l} \sin \frac{\pi ct}{l} + \left(-\frac{bl}{12 \pi c}\right) \sin \frac{3 \pi x}{l} \sin \frac{3 \pi ct}{l} \]. \quad \text{Ans.} \]

**Example 11:** Solve the equation \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} - 2 \frac{\partial^2 u}{\partial y^2} = 0 \) using the transformation \( v = x + y \), \( z = 2x - y \).

[U.P.T.U. (C.O.) 2005]

**Solution:** We have

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} - 2 \frac{\partial^2 u}{\partial y^2} = 0 \] …(1)

Given that

\[ v = x + y \] …(2)

and

\[ z = 2x - y \] …(3)

Now

\[ \frac{\partial u}{\partial x} = \frac{\partial u}{\partial v} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial x} = \frac{\partial u}{\partial v} + 2 \frac{\partial u}{\partial z} \quad \text{using (2) and (3)} \]

\[ \Rightarrow \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial v} + 2 \frac{\partial}{\partial z} \]

\[ \therefore \quad \frac{\partial^2 u}{\partial x^2} = \left( \frac{\partial}{\partial v} + 2 \frac{\partial}{\partial z} \right) \left( \frac{\partial u}{\partial v} + 2 \frac{\partial u}{\partial z} \right) = \frac{\partial^2 u}{\partial v^2} + 4 \frac{\partial^2 u}{\partial z \partial v} + 4 \frac{\partial^2 u}{\partial z^2} \]
and

\[ \frac{\partial u}{\partial y} = \frac{\partial u}{\partial v} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial y} = \frac{\partial u}{\partial y} - \frac{\partial u}{\partial z} \]

\[ \Rightarrow \frac{\partial}{\partial y} = \frac{\partial}{\partial v} - \frac{\partial}{\partial z} \]

\[ \therefore \frac{\partial^2 u}{\partial y^2} = \left( \frac{\partial}{\partial v} - \frac{\partial}{\partial z} \right) \left( \frac{\partial u}{\partial v} - \frac{\partial u}{\partial z} \right) = \frac{\partial^2 u}{\partial v^2} - 2 \frac{\partial^2 u}{\partial z \partial v} + \frac{\partial^2 u}{\partial z^2} \]

Also

\[ \frac{\partial^2 u}{\partial x \partial y} = \left( \frac{\partial}{\partial v} + 2 \frac{\partial}{\partial z} \right) \left( \frac{\partial u}{\partial v} - \frac{\partial u}{\partial z} \right) = \frac{\partial^2 u}{\partial v^2} - 2 \frac{\partial^2 u}{\partial z \partial v} - 2 \frac{\partial^2 u}{\partial z^2} \]

Putting the values of \( \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y} \) and \( \frac{\partial^2 u}{\partial y^2} \) in equation (1), we get

\[ \frac{\partial^2 u}{\partial v^2} + 4 \frac{\partial^2 u}{\partial z \partial v} + 4 \frac{\partial^2 u}{\partial z^2} - 4 \frac{\partial^2 u}{\partial v \partial z} - 2 \frac{\partial^2 u}{\partial z^2} - 2 \frac{\partial^2 u}{\partial v \partial z} - 4 \frac{\partial^2 u}{\partial z \partial v} = 0 \]

\[ \Rightarrow 2 \frac{\partial^2 u}{\partial v^2} - 2 \frac{\partial^2 u}{\partial z^2} + 9 \frac{\partial^2 u}{\partial z \partial v} + 4 \frac{\partial^2 u}{\partial z^2} - 4 \frac{\partial^2 u}{\partial z^2} = 0 \]

\[ \Rightarrow 9 \frac{\partial^2 u}{\partial z \partial v} = 0 \]

\[ \Rightarrow \frac{\partial^2 u}{\partial z \partial v} = 0 \]

Integrating the above equation w.r.t. ‘z’ partially, we get

\[ \frac{\partial u}{\partial v} = f_1(v) \]

Again integrate w.r.t. ‘v’ partially, we get

\[ u = \int f_1(v) \, dv + F_2(z) \]

\[ = F_1(v) + F_2(z) \]

\[ \Rightarrow u = F_1(x+y) + F_2(2x-y). \quad \text{Ans.} \]
Example 12: The points of trisection of a string are pulled aside through the same distance on opposite sides of the position of equilibrium and the string is released from rest. Derive an expression for the displacement of the string at subsequent time and show that the mid-point of the string always remains at rest.

(A.M.I.E.T.E., 1999)

Solution: Let the string OA be trisected at B and C.

Let the equation of vibrating string is

\[ \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \]  \hfill ...(1)

The solution of equation (1) is

\[ y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt) \]  \hfill ...(2)

Now using the boundary conditions

(i) \( y(0, t) = 0 \) \hspace{0.5cm} (ii) \( y(l, t) = 0 \)

Making use of (i) and (ii) in equation (2), we get

\[ y(x, t) = c_2 \sin px(c_3 \cos cpt + c_4 \sin cpt) \]  \hfill ...(3)

where \( p = \frac{n\pi}{l} \).

Next equation of OB' is \( y = \frac{ax}{b/3} \) \( \Rightarrow \ y = \frac{3a}{l} x \)

Equation of B'C' is \( y - a = \frac{a + a}{l - 2l} \left( x - \frac{l}{3} \right) \) \( \Rightarrow \ y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1) \)

\( \Rightarrow \ y = \frac{3a}{l}(l - 2x) \)
and equation of \( C'A \) is
\[
y - 0 = \frac{-a - 0}{2l} (x - l)
\]
\[
\Rightarrow y = \frac{3a}{l} (x - l)
\]
\[\therefore \] The initial conditions of given problem are

(iii) \( \left( \frac{\partial y}{\partial t} \right)_{t = 0} = 0 \)

(iv) \( y(x, 0) = \begin{cases} \frac{3a}{l} x, & 0 \leq x \leq l/3 \\
\frac{3a}{l} (l - 2x), & \frac{l}{3} \leq x \leq \frac{2l}{3} \\
\frac{3a}{l} (x - l), & \frac{2l}{3} \leq x \leq l 
\end{cases} \)

From equation (3), we get
\[
\frac{\partial y}{\partial t} = c_2 \sin px(-c_3 \sin cp t + c_4 \cos cp t)
\]

Using (iii) initial condition in above, we obtain
\[
0 = c_2 \sin px(c_4 cp) \Rightarrow c_4 = 0.
\]

Again from (3), we have
\[
y(x, t) = c_2 c_3 \sin \frac{n\pi x}{l} \cdot \cos \frac{n\pi ct}{l}
\]
\[\therefore \] The general solution of equation (1) is
\[
y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot \cos \frac{n\pi ct}{l} \quad ...(4)
\]

Using (iv) condition in equation (4), we get
\[
.y(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}
\]
\[\therefore \]
\[
b_n = \frac{2}{l} \int_{0}^{l} y(x, 0) \cdot \sin \frac{n\pi x}{l} dx
\]
\[
= \frac{2}{l} \left[ \int_{0}^{l/3} \frac{3a}{l} x \sin \frac{n\pi x}{l} dx + \int_{l/3}^{2l/3} \frac{3a}{l} (l - 2x) \sin \frac{n\pi x}{l} dx + \int_{2l/3}^{l} \frac{3a}{l} (x - l) \sin \frac{n\pi x}{l} dx \right]
\]
\[ y(x, t) = \sum_{n=2}^{\infty} \frac{36a}{n^2 \pi^2} \sin \left( \frac{n\pi x}{3} \right) \sin \left( \frac{n\pi t}{l} \right) \cos \left( \frac{n\pi x}{l} \right). \quad \text{Ans.} \]

**Example 13:** A tightly stretched string with fixed end points \( x = 0 \) and \( x = \pi \) is initially at rest in its equilibrium position. If it is set vibrating by giving to each of its points an initial velocity

\[ \left( \frac{\partial y}{\partial t} \right)_{t=0} = 0.05 \sin x - 0.06 \sin 2x \]

then find the displacement \( y(x, t) \) at any point of string at any time \( t \).

**Solution:** We know that

\[ \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{...(1)} \]
and its solution is

\[ y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt) \]  

...(2)

The boundary conditions are

\( (a) \quad y(0, t) = 0 \) \quad \( (b) \quad y(\pi, t) = 0 \)

Using \( (a) \) in equation (2), we get \( c_1 = 0 \)

\[ y(x, t) = c_2 \sin px(c_3 \cos cpt + c_4 \sin cpt) \]  

...(3)

Using \( (b) \) in equation (3), we get

\[ 0 = c_2 \sin pm(c_3 \cos cpt + c_4 \sin cpt) \]

\[ \Rightarrow \quad \sin pn = 0 = \sin n\pi \quad \Rightarrow \quad p = n. \]

Now, the initial conditions are

\( (c) \quad y(x, 0) = 0 \) \quad \( (d) \quad \left( \frac{\partial y}{\partial t} \right)_{t=0} = 0.05 \sin x - 0.06 \sin 2x \)

At \( t = 0, \) from (3), we have

\[ 0 = c_2c_3 \sin px \quad \Rightarrow \quad c_3 = 0. \]

\[ \therefore \quad \text{Again from (3), we obtain} \]

\[ y(x, t) = c_2c_4 \sin nx \sin cct \quad | \text{As } p = n \]

The general solution is

\[ y(x, t) = \sum_{n=1}^{\infty} b_n \sin nx \sin cct \]  

...(4)

\[ \Rightarrow \quad \frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} b_n \sin nx \cdot (nc \cos cct) \]

At \( t = 0 \)

\[ \left( \frac{\partial y}{\partial t} \right)_{t=0} = 0.05 \sin x - 0.06 \sin 2x = \sum_{n=1}^{\infty} nc \cdot b_n \sin nx \]

\[ \Rightarrow \quad 0.05 \sin x - 0.06 \sin 2x = cb_1 \sin x + 2cb_2 \sin 2x + \cdots \]

\[ \Rightarrow \quad cb_1 = 0.05 \quad \Rightarrow \quad b_1 = \frac{0.05}{c} \quad \text{and} \quad 2cb_2 = -0.06 \quad \Rightarrow \quad b_2 = -\frac{0.03}{c} \]

\[ \therefore \quad \text{From (4), we get} \]

\[ y(x, t) = \frac{0.05}{c} \sin x \sin cct - \frac{0.03}{c} \sin 2x \sin 2ct. \quad \text{Ans.} \]
Example 14: If the string of length \( l \) is initially at rest in equilibrium position and each of its points is given the velocity
\[
v_0 \sin \left( \frac{3\pi x}{l} \right) \cos \left( \frac{2\pi x}{l} \right)
\]
where \( 0 < x < l \) at \( t = 0 \) determine the displacement function \( y(x, t) \).

Solution: The displacement \( y(x, t) \) given by wave equation
\[
\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}
\]
we know that the solution of (1) is
\[
y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)
\]
Using the boundary conditions

(i) \( y(0, t) = 0 \)   (ii) \( y(l, t) = 0 \)

We get from (2)
\[
y(x, t) = c_2 \sin px(c_3 \cos cpt + c_4 \sin cpt)
\]
where \( p = \frac{n\pi}{l} \).

and the initial conditions are

(iii) \( y(x, 0) = 0 \)   (iv) \( \frac{\partial y}{\partial t} \bigg|_{t=0} = v_0 \sin \left( \frac{3\pi x}{l} \right) \cos \left( \frac{2\pi x}{l} \right) \)

Using (iii) in equation (3), we get
\[
0 = c_2 c_3 \sin px \cos cpt \quad \Rightarrow \quad c_3 = 0.
\]
Making use \( c_3 = 0 \) in equation (3), we get
\[
y(x, t) = c_2 c_4 \sin px \sin cpt
\]
where \( p = \frac{n\pi}{l} \).

or the general form of solution is
\[
y(x, t) = \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi x}{l} \right) \sin \left( \frac{n\pi t}{l} \right)
\]

Differentiating partially w.r.t. ‘\( t \)’ equation (4), we get
\[
\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} b_n \left( \frac{n\pi c}{l} \right) \sin \left( \frac{n\pi x}{l} \right) \cos \left( \frac{n\pi t}{l} \right)
\]
\[
\therefore \quad \left( \frac{\partial y}{\partial t} \right) \bigg|_{t=0} = v_0 \sin \left( \frac{3\pi x}{l} \right) \cos \left( \frac{2\pi x}{l} \right) = \sum_{n=1}^{\infty} b_n \left( \frac{n\pi c}{l} \right) \sin \left( \frac{n\pi x}{l} \right)
\[ \Rightarrow \frac{v_0}{2} \left[ \sin \frac{5\pi x}{l} + \sin \frac{\pi x}{l} \right] = \sum_{n=1}^{\infty} b_n \left( \frac{n\pi c}{l} \right) \sin \frac{n\pi x}{l} \]

Equating the coefficient of like terms, we have

\[ \frac{v_0}{2} = b_1 \left( \frac{\pi c}{l} \right) \Rightarrow b_1 = \frac{lv_0}{2\pi} \]

\[ \frac{v_0}{2} = b_2 \left( \frac{5\pi c}{l} \right) \Rightarrow b_2 = \frac{lv_0}{5\pi} \]

and \( b_2 = b_3 = b_4 = b_5 = b_6 = \cdots = 0 \)

Using these values in equation (4), we get the required solution

\[ y(x, t) = \left( \frac{lv_0}{2\pi} \right) \sin \left( \frac{\pi x}{l} \right) \sin \left( \frac{\pi ct}{l} \right) + \left( \frac{lv_0}{5\pi} \right) \sin \left( \frac{5\pi x}{l} \right) \sin \left( \frac{5\pi ct}{l} \right). \quad \text{Ans.} \]

### 5.4 D’ALEMBERT’S SOLUTION OF WAVE EQUATION

Transform the equation \( \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \) to its normal form using the transformation \( u = x + ct, \ v = x - ct \)

and hence solve it. Show that the solution may be put in the form \( y = \frac{1}{2} [f(x + ct) + f(x - ct)] \)

Assume initial conditions \( y = f(x) \) and \( \frac{\partial y}{\partial t} = 0 \) at \( t = 0. \quad [U.P.T.U., \ 2003] \)

**Proof:** Consider one dimensional wave equation

\[ \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \ldots (1) \]

Let \( u = x + ct \) and \( v = x - ct \), be a transformation of \( x \) and \( t \) into \( u \) and \( v \).

then

\[ \frac{\partial y}{\partial x} = \frac{\partial y}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial y}{\partial v} \frac{\partial v}{\partial x} = \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v} \quad \mid \frac{\partial u}{\partial x} = 1, \frac{\partial v}{\partial x} = 1 \]

or

\[ \frac{\partial}{\partial x} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \]

\[ \Rightarrow \quad \frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial y}{\partial x} \right) = \left( \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \right) \left( \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v} \right) \]

\[ = \frac{\partial^2 y}{\partial u^2} + \frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2} + \frac{\partial^2 y}{\partial v \partial u} \]
\[ \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial u^2} + 2 \frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2} \] ...

and
\[ \frac{\partial y}{\partial t} = \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v} = c \frac{\partial y}{\partial u} - c \frac{\partial y}{\partial v} \]
\[ \frac{\partial y}{\partial t} = c, \quad \frac{\partial v}{\partial t} = -c \]

or
\[ \frac{\partial}{\partial t} = c \left( \frac{\partial}{\partial u} - \frac{\partial}{\partial v} \right) \]
\[ \therefore \frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial t} \left( \frac{\partial}{\partial u} - \frac{\partial}{\partial v} \right) = c \left( \frac{\partial}{\partial u} - \frac{\partial}{\partial v} \right) \left( c \frac{\partial y}{\partial u} - c \frac{\partial y}{\partial v} \right) \]
\[ = c^2 \left( \frac{\partial^2 y}{\partial u^2} - \frac{\partial^2 y}{\partial u \partial v} - \frac{\partial^2 y}{\partial v \partial u} + \frac{\partial^2 y}{\partial v^2} \right) \]
\[ \Rightarrow \frac{\partial^2 y}{\partial t^2} = c^2 \left( \frac{\partial^2 y}{\partial u^2} - 2 \frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2} \right) \]

Making use of equations (2) and (3) in equation (1), we get
\[ c^2 \left( \frac{\partial^2 y}{\partial u^2} - 2 \frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2} \right) = c^2 \left( \frac{\partial^2 y}{\partial u^2} + 2 \frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2} \right) \]
\[ \Rightarrow 4c^2 \frac{\partial^2 y}{\partial u \partial v} = 0 \Rightarrow \frac{\partial^2 y}{\partial u \partial v} = 0 \]

Integrating equation (4), w.r.t. ‘v’, we get
\[ \frac{\partial^2 y}{\partial u} = \phi(u) \]

where \( \phi(u) \) is a constant in respect of v.

Again integrate equation (5) w.r.t. ‘u’, we get
\[ y = \int \phi(u) du + \phi_2(v) \]
\[ \Rightarrow y = \phi_1(u) + \phi_2(v) \]
\[ \Rightarrow y(x,t) = \phi_1(x + ct) + \phi_2(x - ct). \]

The solution (6) is D’Alembert’s solution of wave equation

Now, we applying initial conditions \( y = f(x) \) and \( \frac{\partial y}{\partial t} = 0 \) at \( t = 0 \)
From (6), we get at $t = 0$

$$f(x) = \phi_1(x) + \phi_2(x) \quad \ldots(7)$$

and

$$\frac{\partial y}{\partial t} = c\phi'_1(x + ct) - c\phi'_2(x - ct)$$

$$\Rightarrow \quad \left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 = c\phi'_1(x + 0) - c\phi'_2(x - 0)$$

$$\Rightarrow \quad \phi'_1(x) - \phi'_2(x) = 0$$

$$\Rightarrow \quad \phi'_1(x) = \phi'_2(x)$$

On integrating, we get

$$\phi_1(x) = \phi_2(x) + c_1 \quad \ldots(8)$$

Using equation (8) in equation (7), we get

$$f(x) = \phi_2(x) + c_1 + \phi_2(x) = 2\phi_2(x) + c_1$$

$$\Rightarrow \quad \phi_2(x) = \frac{1}{2}\left[f(x) - c_1\right] \quad \Rightarrow \quad \phi_2(x - ct) = \frac{1}{2}\left[f(x - ct) - c_1\right]$$

and

$$\phi_1(x) = \frac{1}{2}\left[f(x) + c_1\right] \quad \Rightarrow \quad \phi_1(x + ct) = \frac{1}{2}\left[f(x + ct) + c_1\right]$$

Putting the values of $\phi_1(x + ct)$ and $\phi_2(x - ct)$ in equation (6), we get

$$y(x, t) = \frac{1}{2}\left[f(x + ct) + f(x - ct)\right]. \quad \text{Proved.}$$

**EXERCISE 5.1**

1. Solve completely the equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$, representing the vibrations of a string of length $l$, fixed at both ends, given that $y(0, t) = 0, y(l, t) = 0; y(x, 0) = f(x)$ and $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$,

$$0 < x < l. \quad \text{Ans.} \quad y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}; \quad b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx \quad (U.P.T.U. \ 2005)$$

2. A taut string of length $2l$ is fastened at both ends. The midpoint of the string is taken to a height $h$ and then released from rest in that position. Find the displacement of the string.

$$\text{Ans.} \quad y(x, t) = \frac{8h}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n - 1)^2} \sin \left(\frac{(2n-1)\pi x}{2l}\right) \cos \left(\frac{(2n-1)\pi ct}{2l}\right)$$
3. A uniform elastic string of length 60 cm is subjected to a constant tension of 2 kg. If the ends are fixed and the initial displacement is \(y(x, 0) = 60x - x^2\) for \(0 < x < 60\) while the initial velocity is zero, find \(y(x, t)\).

\[
\text{Ans. } y(x, t) = \frac{8 \times 60^2}{\pi^3} \left( \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin \left( \frac{(2n-1)\pi x}{60} \right) \cos \left( \frac{(2n-1)\pi ct}{60} \right) \right)
\]

4. A taut string of length 20 cms fastened at both ends is displaced from its position of equilibrium by imparting to each of its points an initial velocity is given by

\[
v = \begin{cases} 
  x & \text{in } 0 < x < 10 \\
  20 - x & \text{in } 10 < x < 20
\end{cases}
\]

where \(x\) is the distance from one end. Determine the displacement at any subsequent time.

\[
\text{Ans. } y(x, t) = \frac{1600}{\pi^3 c^3} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin \left( \frac{n\pi x}{20} \right) \sin \left( \frac{n\pi ct}{20} \right)
\]

5. A tightly stretched string with fixed end points \(x = 0\) and \(x = \pi\) is initially at rest in the equilibrium position. If it is set vibrating by giving each point a velocity \(\left( \frac{\partial y}{\partial t} \right)_{x=0} = 0.03 \sin x - 0.04 \sin 3x\) then find the displacement \(y(x, t)\) at any point of the string at any time \(t\).

\[
\text{Ans. } y(x, t) = \frac{1}{c} [0.03 \sin x \sin ct - 0.133 \sin 3x \sin 3ct]
\]

6. A tightly stretched string with fixed end points \(x = 0\) and \(x = l\) is initially in a position given by \(y = y_0 \sin^3 \left( \frac{\pi x}{l} \right)\). If it is released from rest from this position, find the displacement \(y(x, t)\).

\[
\text{Ans. } y(x, t) = \frac{y_0}{4} \left( 3 \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l} - \sin \frac{3\pi x}{l} \cos \frac{3\pi ct}{l} \right)
\]

7. A tightly stretched flexible string has its ends fixed at \(x = 0\) and \(x = l\). At time \(t = 0\) the string is given a shape defined by \(y = \mu x (l - x)\), \(\mu\) is a constant and then released. Find the displacement \(y(x, t)\) of any point \(x\) of the string at any time \(t > 0\).

\[
\text{Ans. } y(x, t) = \frac{4\mu l^2}{\pi^3} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^3} \cos \frac{n\pi x}{l} \sin \frac{n\pi ct}{l}
\]

8. A uniform string of length \(l\) is struck in such a way that an initial velocity of \(v_0\) is imparted to the portion of the string \(\frac{l}{4}\) and \(\frac{3l}{4}\) while the string is in equilibrium position. Find the
subsequent displacement of the string as a function of $x$ and $t$.

\[
\text{Ans. } y(x, t) = \frac{4lv_0}{\sqrt{2\pi^2c}} \left( \frac{1}{1^2} \sin \frac{\pi x}{l} \sin \frac{\pi ct}{l} - \frac{1}{3^2} \sin \frac{3\pi x}{l} \sin \frac{3\pi ct}{l} + \frac{1}{5^2} \sin \frac{5\pi x}{l} \sin \frac{5\pi ct}{l} \ldots \right)
\]

9. Find the displacement if a string of length $a$ is vibrating between fixed end points with initial velocity zero and initial displacement given by $y(x, 0) = \begin{cases} \frac{2px}{a} & \text{for } 0 < x < \frac{a}{2} \\ 2p - \frac{2px}{a} & \text{for } \frac{a}{2} < x < a. \end{cases}$

\[
\text{Ans. } y(x, t) = \frac{8p}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin \frac{(2n-1)\pi x}{a} \cos \frac{(2n-1)\pi ct}{a}
\]

10. A taut string of length $l$ has its ends $x = 0$ and $x = l$ fixed. The mid-point is taken to a small height $h$ and released from rest at time $t = 0$. Find the displacement $y(x, t)$.

\[
\text{Ans. } y(x, t) = \sum_{n=1}^{\infty} \frac{8h}{n^2\pi^2} \sin \frac{n\pi x}{2l} \cos \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}
\]

11. If a string of length $l$ is released from rest in the position $y = \frac{4kx(l-x)}{l^2}$ show that the motion is described by the equation.

\[
\text{Ans. } y(x, t) = \frac{32k}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} \sin \frac{(2n+1)\pi x}{l} \cos \frac{(2n+1)\pi ct}{l}
\]

### 5.5 ONE DIMENSIONAL FLOW

In this section we set up the mathematical model for one dimensional heat flow and derive the corresponding partial differential equation.

Consider a bar or a rod of equal thickness at every point.

Let the area of cross-sectional = $A$ cm$^2$.

and density of material of rod = $\rho$ gr/cm$^3$

Here we consider a small element $PQ$ of length $\delta x$. 
The mass of the element $PQ = A \rho \delta x$

Let $u(x, t)$ is the temperature of the rod at a distance $x$ at time $t$.

We know that the amount of heat in a body is always proportional to the mass of the body and to the temperature change.

Thus the rate of increase of heat in element

$$= sA \rho \delta x \frac{\partial u}{\partial t} \quad (s \text{ is specific heat}) \quad \ldots (i)$$

Since the direction of heat flow in a body becomes always toward decreasing temperature. Physical experiment shows that the rate of flow is proportional to the area and to the temperature gradient normal to the area. If we suppose $Q_1$ and $Q_2$ are the quantities of heat flowing at the points $P$ and $Q$ respectively,

then

$$Q_1 = -kA \left( \frac{\partial u}{\partial x} \right)_{x + \delta x} \quad \text{per second} \quad \text{The negative sign shows the direction of heat flow towards lower temperature}$$

and

$$Q_2 = -kA \left( \frac{\partial u}{\partial x} \right)_{x} \quad \text{per second}$$

where $k$ is a constant known as thermal conductivity.

$$\therefore \text{Total amount of heat in the element} = Q_1 - Q_2 = kA \left[ \left( \frac{\partial u}{\partial x} \right)_{x + \delta x} - \left( \frac{\partial u}{\partial x} \right)_{x} \right] \quad \text{per second} \quad \ldots (ii)$$

From $(i)$ and $(ii)$

$$sA \rho \delta x \frac{\partial u}{\partial t} = kA \left[ \left( \frac{\partial u}{\partial x} \right)_{x + \delta x} - \left( \frac{\partial u}{\partial x} \right)_{x} \right]$$

$$\Rightarrow \quad \frac{\partial u}{\partial t} = \frac{k}{\rho s} \left[ \frac{\partial u}{\partial x} \right]_{x + \delta x} - \frac{\partial u}{\partial x} \right]_{x}$$

Taking the limit as $\delta x \to 0$ i.e., when $x + \delta x \to x$

$$\therefore \quad \frac{\partial u}{\partial t} = \frac{k}{\rho s} \lim_{\delta x \to 0} \left[ \frac{\partial u}{\partial x} \right]_{x + \delta x} - \frac{\partial u}{\partial x} \right]_{x}$$

$$= \frac{k}{\rho s} \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) = \frac{k}{\rho s} \frac{\partial^2 u}{\partial x^2}$$
Let \( \frac{k}{\rho s} = c^2 \) is called diffusivity of the substance

Thus

\[
\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}.
\]

Alternate: Since the rate of flow is proportional to the gradient of the temperature

\[
\vec{V} = -k \nabla u
\]

where \( \vec{V} \) is the velocity of the heat flow and \( u(x, y, z, t) \) is temperature and \( k \) is thermal conductivity.

Let \( R \) be the region in the body and let \( s \) be its boundary surface.

Then amount of heat leaving \( R \) per unit time = \( \iint_S \vec{V} \cdot \hat{n} \, dA \)

where \( \hat{n} \) is the outward unit normal of \( s \).

From (i) and by Gauss divergence theorem, we obtain

\[
\iint_S \vec{V} \cdot \hat{n} \, dA = -k \iiint_R \nabla \cdot (\nabla u) \, dx \, dy \, dz
\]

\[
= -k \iiint_R \nabla^2 u \, dx \, dy \, dz \tag{ii}
\]

And the amount of heat in \( R \) is given by

\[
Q = \iiint_R \rho u \, dx \, dy \, dz
\]

where \( \rho \) is the specific heat of the material of the body. Here the rate of decrease of \( Q \) is given by

\[
- \frac{\partial Q}{\partial t} = - \iiint_R \rho \frac{\partial u}{\partial t} \, dx \, dy \, dz \tag{iii}
\]

Since the rate of decrease of \( Q \) must be equal to the amount of heat leaving \( R \), from (ii) and (iii), we get

\[
- \iiint_R \rho \frac{\partial u}{\partial t} \, dx \, dy \, dz = -k \iiint_R \nabla^2 u \, dx \, dy \, dz
\]

\[
\Rightarrow \iiint_R \left( \rho \frac{\partial u}{\partial t} - k \nabla^2 u \right) \, dx \, dy \, dz = 0
\]

Since this holds for any region \( R \) in the body the integrand must be zero everywhere.

\[
\therefore \quad \rho \frac{\partial u}{\partial t} = k \nabla^2 u
\]

\[
\Rightarrow \quad \frac{\partial u}{\partial t} = \frac{k}{\rho} \nabla^2 u \tag{iv}
\]
Let \( \frac{k}{s \rho} = c^2 \), then equation (iv) reduces in the form

\[
\frac{\partial u}{\partial t} = c^2 \nabla^2 u \quad \Rightarrow \quad \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}.
\]

\[\nabla^2 u = \nabla \cdot \nabla u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\]

in one dimension i.e., along \( X \).

\[\nabla^2 u = \frac{\partial^2 u}{\partial x^2}\]

5.5.1 Solution of One Dimensional Heat Equation \( (U.P.T.U. \ 2007) \)

We know that the heat equation

\[
\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{...(1)}
\]

Let \( u(x, t) = X(x) T(t) \) \( \quad \text{...(2)} \)

\[
\Rightarrow \quad \frac{\partial u}{\partial x} = T \frac{dX}{dx} \quad \text{or} \quad \frac{\partial^2 u}{\partial x^2} = T \frac{d^2 X}{dx^2}
\]

\[
\frac{\partial u}{\partial t} = X \frac{dT}{dt}
\]

Using these values in equation (1), we get

\[
X \frac{dT}{dt} = c^2 T \frac{d^2 X}{dx^2}
\]

\[
\Rightarrow \quad \frac{1}{c^2 T} \frac{dT}{dt} = \frac{1}{X} \frac{d^2 X}{dx^2} = k \quad \text{...(3)}
\]

Taking 2nd and 3rd terms

\[
\therefore \quad \frac{1}{X} \frac{d^2 X}{dx^2} = k \quad \Rightarrow \quad \frac{d^2 X}{dx^2} - kX = 0
\]

\[
\Rightarrow \quad (D^2 - k) X = 0
\]

Hence

\[X = c_1 e^{\sqrt{k} x} + c_2 e^{-\sqrt{k} x}\]

and

\[
\frac{1}{c^2 T} \frac{dT}{dt} = k \quad \Rightarrow \quad \frac{dT}{T} = kc^2 dt
\]

On integrating,

\[\log_e T = kc^2 t + \log c_3\]

\[
\Rightarrow \quad T = c_3 e^{kc^2 t}
\]
\[ u(x, t) = (c_1 e^{px} + c_2 e^{-px}) e^{-\frac{c^2}{2} t} \]

There are arise following cases:

**Case I:** If \( k > 0 \), let \( k = p^2 \)

then

\[ u(x, t) = (c_1 e^{pt} + c_2 e^{-pt}) e^{\frac{p^2 t}{2}}. \]  ... (A)

**Case II:** If \( k < 0 \), Let \( k = -p^2 \)

then A.E. is \( m^2 = -p^2 \Rightarrow m = \pm pi \)

\[ X = c_1 \cos px + c_2 \sin px \]

then

\[ u(x, t) = (c_1 \cos px + c_2 \sin px) e^{-\frac{p^2 t}{2}}. \]  ... (B)

**Case III:** If \( k = 0 \), then

\[ u(x, t) = (c_1 + c_2 x) c_3. \]  ... (C)

Since the physical nature of the problem is periodic so the suitable solution of the heat equation is

\[ u(x, t) = (c_1 \cos px + c_2 \sin px) e^{-\frac{p^2 t}{2}}. \]  ... (4)

The boundary conditions are

(i) \( u(0, t) = 0 \) and (ii) \( u(l, t) = 0 \)

and initial condition is (iii) \( u(x, 0) = f(x) \).

Using (i) boundary condition in (4), we get

\[ 0 = c_1 e^{-\frac{p^2 l}{2} t} \Rightarrow c_1 = 0 \]

\[ \therefore \text{From (4), we get} \]

\[ u(x, t) = c_2 c_3 \sin px e^{-\frac{p^2 t}{2}} \]  ... (5)

Using (ii) boundary condition in (5), we get

\[ 0 = c_2 c_3 \sin pl \Rightarrow \sin pl = 0 = \sin n\pi \Rightarrow \boxed{p = \frac{n\pi}{l}} \]

\[ \therefore \]

\[ u(x, t) = c_2 c_3 \sin \frac{n\pi x}{l} e^{-\frac{n^2 \pi^2 c^2 t}{l}} \]

The general form of above solution is

\[ u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{-\frac{n^2 \pi^2 c^2 t}{l}} \]  \( (b_n = c_2 c_3) \)  ... (6)
Again using initial condition (iii) in equation (6), we get

\[ f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \]

which represents Fourier half range sine series so, we have

\[ b_n = \frac{2}{l} \int_{0}^{l} f(x) \sin \frac{n\pi x}{l} \, dx \]

Thus the required solution is

\[ u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot e^{-\frac{n^2 \pi^2 c^2 t}{l}} \]

where \( b_n = \frac{2}{l} \int_{0}^{l} f(x) \sin \frac{n\pi x}{l} \, dx \).

**Remark:** In steady state \( \frac{\partial u}{\partial t} = 0 \), so \( \frac{\partial^2 u}{\partial x^2} = 0 \).

**Example 15:** Determine the solution of one dimension heat equation

\[ \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \]

Under the conditions \( u(0, t) = u(l, t) = 0 \) and \( u(x,0) = \begin{cases} x & \text{if } 0 \leq x \leq l/2 \\ l-x & \text{if } l/2 \leq x \leq l. \end{cases} \)

**Solution:** We have

\[ \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{...(1)} \]

We know that the solution of equation (1) (on page 470, equation 4) is

\[ u(x, t) = (c_1 \cos px + c_2 \sin px) c_3 e^{-\frac{c^2}{l} x^2 t} \quad \text{...(2)} \]

At \( x = 0 \), we get

\[ 0 = c_1 c_3 e^{-\frac{c^2}{l} 0^2 t} \Rightarrow c_1 = 0. \]

\[ \therefore \text{ From equation (2), we get } u(x,t) = c_2 c_3 \sin px \cdot e^{-\frac{c^2}{l} x^2 t} \quad \text{...(3)} \]
Again at \( x = l \) from (3), we get
\[
0 = c_2 c_3 \sin pl \cdot e^{-\frac{n^2 \pi^2 c_1 t}{l}} \quad \Rightarrow \quad \sin pl = \sin n\pi
\]
\[
\Rightarrow \quad p = \frac{n\pi}{l}
\]
From (3), we get
\[
u(x, t) = c_2 c_3 \sin \frac{n\pi x}{l} e^{-\frac{n^2 \pi^2 c_1 t}{l}}
\]
\[
\Rightarrow \quad \text{Therefore the general form of solution can be written as}
\]
\[
u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{-\frac{n^2 \pi^2 c_1 t}{l}}
\]
\[
\text{...(4)}
\]
At \( t = 0 \), from equation (4), we get
\[
u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}
\]
\[
\therefore \quad b_n = \frac{2}{l} \int_{0}^{l} u(x, 0) \sin \frac{n\pi x}{l} \, dx
\]
\[
= \frac{2}{l} \left[ \int_{0}^{l/2} x \sin \frac{n\pi x}{l} \, dx + \int_{l/2}^{l} (l-x) \sin \frac{n\pi x}{l} \, dx \right]
\]
\[
= \frac{2}{l} \left[ \left. \int x \sin \frac{n\pi x}{l} \, dx \right|_{0}^{l/2} + \left. \int (l-x) \sin \frac{n\pi x}{l} \, dx \right|_{l/2}^{l} \right]
\]
\[
= \frac{2}{l} \left[ \left. -x \cos \frac{n\pi x}{l} \right|_{0}^{l/2} + \left. \frac{l^2}{n^2 \pi^2} \sin \frac{n\pi x}{l} \right|_{0}^{l/2} \right]
\]
\[
= \frac{2}{l} \left[ -\frac{l}{n \pi} \cos \frac{n\pi x}{l} + \frac{l^2}{n^2 \pi^2} \sin \frac{n\pi x}{l} \right]_{0}^{l/2}
\]
\[
= \frac{2}{l} \left[ -\frac{l}{2n \pi} \cos \frac{n\pi x}{l} + \frac{l^2}{n^2 \pi^2} \sin \frac{n\pi x}{l} - \frac{l^2}{2n \pi} \cos \frac{n\pi x}{l} + \frac{l^2}{2n^2 \pi^2} \sin \frac{n\pi x}{l} \right]
\]
\[
= \frac{4l}{n^2 \pi^2} \sin \left( \frac{n\pi}{2} \right)
\]
\[
\therefore \quad \text{From equation (4), we get}
\]
\[
u(x, t) = \frac{4l}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi x}{l} \sin \frac{n\pi x}{l} e^{-\frac{n^2 \pi^2 c_1 t}{l}}. \quad \text{Ans.}
\]
Example 16: An insulated rod of length $l$ has its ends $A$ and $B$ maintained 0°C and 100°C respectively until steady state conditions prevail. If $B$ is suddenly reduced to 0°C and maintained at 0°C find the temperature at a distance $x$ from $A$ at time $t$. [U.P.T.U., 2004, 2005]

Solution: From one dimensional and equation, we have

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

...(1)

The boundary conditions are

(i) $u(0, t) = 0°C$ and (ii) $u(l, t) = 100°C$

In steady state condition $\frac{\partial u}{\partial t} = 0$ here from (1), we get

$$\frac{\partial^2 u}{\partial x^2} = 0$$

On integrating, we get $u(x) = c_1 x + c_2$ ...

where $c_1$ and $c_2$ are constants to be determined

At $x = 0$, from equation (2), we have

$$0 = c_2.$$ 

and

at $x = l$, $100 = c_1 l + 0 \Rightarrow c_1 = \frac{100}{l}$. 

:. From (2)

$$u(x) = \frac{100}{l} x$$

...(3)

Now the temperature at $B$ is suddenly changed we have again transient state. If $u(x, t)$ is the subsequent temperature function, the boundary conditions are

(iii) $u(0, t) = 0°C$, (iv) $u(l, t) = 0°C$ and the initial condition (v) $u(x, 0) = \frac{100}{l} x$

Since the subsequent steady state function $u_s(x)$ satisfies the equation

$$\frac{\partial^2 u_s}{\partial x^2} = 0$$

or

$$\frac{d^2 u_s}{dx^2} = 0 \Rightarrow u_s(x) = c_3 x + c_4$$

at $x = 0$, we get

$$0 = c_4$$

and at $x = l$, we get

$$0 = c_3 l + 0 \Rightarrow c_3 = 0$$

Thus

$$u_s(x) = 0$$

...(4)
If $u_T(x, t)$ is the temperature in transient state then the temperature distribution in the rod $u(x, t)$ can be expressed in the form

$$u(x, t) = u_s(x) + u_T(x, t)$$

$$\Rightarrow \quad u(x, t) = u_T(x, t) \quad \text{As } u_s(x) = 0 \quad \text{...(5)}$$

Again from heat equation, we have

$$\frac{\partial u_T}{\partial t} = c^2 \frac{\partial^2 u_T}{\partial x^2} \quad \text{...(6)}$$

The solution of equation (6) is

$$u_T(x, t) = (c_1 \cos px + c_2 \sin px) e^{-c^2p^2t}$$

$$\Rightarrow \quad u(x, t) = (c_1 \cos px + c_2 \sin px) e^{-c^2p^2t} \quad \text{...(7)}$$

At $x = 0, u(0, t) = 0$

$$\Rightarrow \quad 0 = c_1 e^{-c^2p^2t} \Rightarrow c_1 = 0.$$ 

From (7), we get

$$u(x, t) = c_2 \sin px \cdot e^{-c^2p^2t} \quad \text{...(8)}$$

Again at $x = l, \quad u (l, t) = 0$

$$\Rightarrow \quad 0 = c_2 c_3 \sin pl \cdot e^{-c^2p^2t} \quad \Rightarrow \quad \sin pl = 0 = \sin n\pi$$

$$\Rightarrow \quad p = \frac{n\pi}{l}$$

From (8), we get

$$u(x, t) = c_2 c_3 \sin \frac{nx}{l} e^{-c^2 \frac{nx}{l}^2}$$

$$\Rightarrow \quad u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{-c^2 \frac{nx}{l}^2} \quad \text{...(9)}$$

Using initial condition i.e., at $t = 0, \quad u = \frac{100}{l} x$, we get

$$u(x, 0) = \frac{100}{l} x = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$\therefore \quad b_n = \frac{2}{l} \int_0^l \frac{100x}{l} \sin \frac{n\pi x}{l} \, dx$$
APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

\[ \frac{\partial u}{\partial t} = \frac{c^2}{l^2} \frac{\partial^2 u}{\partial x^2} \]  

...(1)

We know that the solution of equation (1) is given by

\[ u(x, t) = (c_1 \cos px + c_2 \sin px) e^{-\frac{n^2\pi^2 c^2}{l^2} t} \]  

...(2)

At \( x = 0, u = 0 \)

\[ 0 = c_1 \]  

From (2), we get

\[ u(x, t) = c_2 \frac{n\pi}{l} x e^{-\frac{n^2\pi^2 c^2}{l^2} t} \]  

...(3)

Again at \( x = l, u = 0 \)

\[ 0 = c_2 \frac{n\pi}{l} l e^{-\frac{n^2\pi^2 c^2}{l^2} t} \]

\[ \sin n\pi = 0 \]

\[ p = \frac{n\pi}{l} \]

From (3) the general solution of equation (1) is

\[ u(x, t) = \sum_{n=1}^{\infty} b_n \sin n\pi \frac{x}{l} e^{-\frac{n^2\pi^2 c^2}{l^2} t} \]  

...(4)

At \( t = 0, \) \( u = x \)

\[ x = \sum_{n=1}^{\infty} b_n \sin n\pi \frac{x}{l} \]

Example 17: Determine the solution of one dimensional heat equation \( \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \) subject to the boundary conditions \( u(0, t) = 0, u(l, t) = 0 (t > 0) \) and the initial condition \( u(x, 0) = x, l \) being the length of the bar.

(\textit{U.P.T.U. 2006})

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\pi \frac{x}{l} e^{-\frac{n^2\pi^2 c^2}{l^2} t} \]. \textbf{Ans.}
\[ b_n = \frac{2}{l} \left[ \int_0^l x \sin \frac{n\pi x}{l} \, dx = \frac{2}{l} \left[ \frac{x}{n\pi} \left( -\cos \frac{n\pi x}{l} \right) - \left( -\frac{l^2}{n^2 \pi^2} \sin \frac{n\pi x}{l} \right) \right]_0^l \right] \]

\[ = \frac{2}{l} \left[ \left( \frac{l}{n\pi} (-\cos n\pi) + \frac{l^2}{n^2 \pi^2} \sin n\pi \right) - 0 \right] \]

\[ \Rightarrow \quad b_n = \frac{2}{l} \left[ \frac{-l^2}{n\pi} (-1)^n \right] = (-1)^{n+1} \frac{2l}{n\pi} \]

Putting the value of \( b_n \) in equation (4), we get

\[ u(x, t) = \frac{2l}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \sin \frac{n\pi x}{l} e^{-\frac{n^2 \pi^2 c^2 t}{l^2}}. \text{ Ans.} \]

**Example 18:** A homogeneous rod of conducting material of length \( l \) has its ends kept at zero temperature. The temperature at the centre is \( T \) and falls uniformly to zero at the two ends. Find the temperature distribution.

**Solution:** From heat equation in one dimension, we have

\[ \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}. \quad \ldots(1) \]

The boundary conditions are (i) \( u(0, t) = 0 \) (ii) \( u(l, t) = 0 \).

Since the temperature at the centre is \( T \) and falls uniformly to zero at the two ends, its distribution at \( t = 0 \) is given in the figure.

The equation of line segment \( OB \) is \( u = \frac{2T}{l} x; \ 0 \leq x \leq \frac{l}{2} \) and the equation of line segment \( BA \) is given by
\[ u = \frac{2T(l-x)}{l} \quad \frac{l}{2} \leq x \leq l \]  

Thus initial condition i.e., at \( t = 0 \) is given by

\[
u(x, 0) = \begin{cases} 
\frac{2Tx}{l}, & \text{when } 0 \leq x \leq \frac{l}{2} \\
\frac{2T(l-x)}{l}, & \text{when } \frac{l}{2} \leq x \leq l
\end{cases}
\]

We know that the solution of equation (1) satisfying the boundary conditions (i) and (ii) is given by

\[
u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{-\frac{n^2 \pi^2 c^2 t}{l^2}} \quad \ldots(2)
\]

At \( t = 0 \), from (2), we get

\[
u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \]

\[ \therefore \quad b_n = \frac{2}{l} \int_0^l \nu(x, 0) \cdot \sin \frac{n\pi x}{l} \, dx \]

\[
b_n = \frac{2}{l} \left[ \int_0^{\frac{l}{2}} \frac{2Tx}{l} \sin \frac{n\pi x}{l} \, dx + \int_{\frac{l}{2}}^l \frac{2T(l-x)}{l} \sin \frac{n\pi x}{l} \, dx \right]
\]

\[
= \frac{4T}{l^2} \left[ \left\{ -x \cos \frac{n\pi x}{l} + \left( \frac{l}{n\pi} \right)^2 \sin \frac{n\pi x}{l} \right\}_0^{\frac{l}{2}} + \left\{ -(l-x) \cos \frac{n\pi x}{l} - \left( \frac{l}{n\pi} \right)^2 \sin \frac{n\pi x}{l} \right\}_{\frac{l}{2}}^l \right]
\]

\[
= \frac{4T}{l^2} \left[ -\frac{l^2}{2n\pi} \cos \frac{n\pi}{2} + \frac{l^2}{n^2 \pi^2} \sin \frac{n\pi}{2} + \frac{l^2}{2n\pi} \cos \frac{n\pi}{2} + \frac{l^2}{n^2 \pi^2} \sin \frac{n\pi}{2} \right]
\]

\[
\Rightarrow \quad b_n = \frac{8T}{n^2 \pi^2} \sin \frac{n\pi}{2}
\]

Putting the value of \( b_n \) in equation (2), we get

\[
u(x, t) = \frac{8T}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi x}{l} \sin \frac{n\pi x}{l} e^{-\frac{n^2 \pi^2 c^2 t}{l^2}}. \quad \text{Ans.}
\]
**Example 19:** A rod of length 10 cm has its ends A and B kept at 50°C and 100°C until steady state conditions prevail. The temperature at A suddenly raised to 90°C and that at B is lowered to 60°C and they are maintained. Find the temperature at a distance from one end at time t.

**Solution:** From one dimensional heat equation

\[ \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \]  

...(1)

The boundary conditions are (i) \( u(0, t) = 50^\circ C \) and (ii) \( u(10, t) = 100^\circ C \)

In steady state condition \( \frac{\partial u}{\partial t} = 0 \), so from (1), we get

\[ \frac{\partial^2 u}{\partial x^2} = 0 \]

On integrating, we get \( u(x) = C_1 x + C_2 \)  

...(2)

Making use of boundary conditions in above equation, we get

50 = C₂  
100 = 2C₁ + C₂

\( \therefore \) From equation (2) \( u(x) = 5x + 50 \)  

...(3)

Now, the temperatures at A and B are suddenly changed we have again transient state. If \( u_1(x, t) \) is subsequent temperature function then the boundary conditions are

\( u_1(0, t) = 90^\circ C, \quad u_1(10, t) = 60^\circ C \)

and the initial condition

\( u_1(x, 0) = 5x + 50 \)

If \( u_s(x) \) is the subsequent steady state function then

\[ \frac{\partial^2 u_s}{\partial x^2} = 0 \implies \frac{d^2 u_s}{dx^2} = 0 \]

\[ \therefore u_s(x) = C_3 x + C_4 \]  

...(4)

At \( x = 0 \), we get

\[ 90 = C_4 \text{ and at } x = 10, \text{ we get} \]

\[ 60 = 10C_3 + 90 \implies C_3 = -3 \]

\( \therefore u_s(x) = -3x + 90 \)

Thus the total temperature distribution in the rod at a distance \( x \) at time \( t \) is given by

\[ u(x, t) = u_r(x, t) \]

where \( u_r(x, t) \) is the temperature in transient state

\[ \implies u(x, t) = (-3x + 90) + u_r(x, t) \]  

...(5)
Now again by heat equation, we have
\[ \frac{\partial u_T}{\partial t} = c^2 \frac{\partial^2 u_T}{\partial x^2} \]

We know that the solution of above equation \( u_T(x, t) \) is given by
\[ u_T(x, t) = (C_1 \cos px + C_2 \sin px)C_3 e^{-c^2 p^2 t} \] ....(6)

Now the boundary conditions which satisfied by \( u_T(x, t) \) are as follows:
\[ u_T(0, t) = u_k(0, t) - u_x(0) = 90 - 90 = 0 \]
\[ u_T(l, t) = u_k(10, t) - u_k(10) = 60 - 60 = 0 \]
and
\[ u_T(x, 0) = u_k(x, 0) - u_k(x) \]
\[ \Rightarrow u_T(x, 0) = 5x + 50 - (-3x + 90) = 8x - 40 \]

The general solution for \( u_T(x, t) \) is given by
\[ u_T(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{10} e^{-\frac{n^2 \pi^2 c^2 t}{100}} \] ....(7)

At
\[ t = 0, \quad u_T = 8x - 40 \]
\[ \Rightarrow 8x - 40 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{10} \]
\[ \therefore b_n = \frac{2}{10} \int_{0}^{10} (8x - 40) \sin \frac{n\pi x}{10} \, dx \]
\[ = \frac{1}{5} \left[ -\frac{10}{n\pi} (8x - 40) \cos \frac{n\pi x}{10} + \frac{800}{n^2 \pi^2} \sin \frac{n\pi x}{10} \right]_{0}^{10} \]
\[ = \frac{1}{5} \left[ -\frac{400}{n\pi} \cos n\pi + \frac{800}{n^2 \pi^2} \sin n\pi - \frac{400}{n\pi} \right] \]
\[ \Rightarrow b_n = \frac{1}{5} \left[ -\frac{400}{n\pi} (-1)^n - \frac{400}{n\pi} \right] = \frac{80}{n\pi} \{(-1)^n + 1\} \]
\[ \Rightarrow b_n = \begin{cases} 0, & \text{when } n \text{ is odd} \\ \frac{-160}{n\pi}, & \text{when } n \text{ is even} \end{cases} \]

From (7), we get
\[ u_T(x, t) = \sum_{n=1}^{\infty} \frac{160}{2m\pi} \sin \frac{2m\pi x}{10} e^{-\frac{4m^2 \pi^2 c^2 t}{100}} \quad |n = 2m \text{(even)} \]
or
\[ u_t(x, t) = -\frac{80}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} \sin \frac{m\pi x}{5} e^{-\frac{m^2\pi^2 c^2 t}{25}} \] ...(8)

Hence from equation (5), we get
\[ u(x, t) = -3x + 90 \sum_{m=1}^{\infty} \frac{1}{m} \sin \frac{m\pi x}{5} e^{-\frac{m^2\pi^2 c^2 t}{25}}. \quad \text{Ans.} \]

**Example 20:** The temperature of a bar 50 cm long with insulated sides is kept at 0°C at one end and 100°C at the other end until steady conditions prevail. The two ends are then suddenly insulated so that the temperature gradient is zero at each end thereafter. Find the temperature distribution.

**Solution:** The temperature function \( u(x, t) \) is the solution of the one dimensional heat equation
\[ \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \] ...(1)

When the steady state condition prevails \( \frac{\partial u}{\partial t} = 0 \) and hence from (1), we get
\[ \frac{\partial^2 u}{\partial x^2} = 0 \]

On integrating, we get
\[ u(x) = c_1 x + c_2 \] ...(2)

At \( x = 0, \ u = 0 \)
\[ 0 = c_2 \]

and at \( x = 50, u = 100, \) from (2), we get
\[ 100 = 50 c_1 + 0 \Rightarrow c_1 = 2 \]

Hence
\[ u(x) = 2x \Rightarrow u(x, 0) = 2x \] ...(3)

and the subsequent temperature function \( u_1(x, t) \) satisfy the boundary conditions
\[ u_1(0, t) = 0, \quad u_1(50, t) = 0 \]

Under these conditions, we find the steady state function \( u_s(x) \) vanishes
\[ i.e., \quad u_s(x) = 0 \quad \text{and} \quad \left. u_t \rightarrow u_s \right|_{u_t \rightarrow u_s} \]

\[ \Rightarrow \quad u(x, t) = u_s(x) + u_f(x, t) = 0 + u_f(x, t) \]

\[ \Rightarrow \quad u(x, t) = u_f(x, t) \] ...(4)

where \( u_f(x, t) \) is the temperature in transient state which satisfied the boundary conditions
\[ u_f(0, t) = 0 = u_f(50, t) \]

\[ \therefore \) The temperature \( u_f(x, t) \) can be obtained by the solution of one dimensional heat equation
\[ u_f(x, t) = (c_1 \cos px + c_2 \sin px) c_3 e^{-\frac{p^2}{2r^2}} \] ...(5)
At \( x = 0, \ u_T = 0 \)
\[
\Rightarrow \quad 0 = c_1 c_3 e^{-c^2 c^2 t} \quad \Rightarrow \quad c_1 = 0 \quad \text{(otherwise } u_T(x, t) = 0)\]
From (5), we get
\[
u_T(x, t) = c_2 c_3 \sin px \cdot e^{-c^2 c^2 t} \quad \ldots (6)
\]
And at \( x = 50, \ u_T = 0 \)
\[
\Rightarrow \quad 0 = c_2 c_3 \sin 50p \cdot e^{-c^2 c^2 t}
\]
\[
\Rightarrow \quad \sin 50p = 0 = \sin n\pi \quad \Rightarrow \quad p = \frac{n\pi}{50}
\]
\[
\therefore \quad \text{From (6), we get}
\]
\[
u_T(x, t) = c_2 c_3 \sin \frac{n\pi x}{50} e^{-n^2 c^2 t} \quad \frac{2500}{2500}
\]
The general form of solution is
\[
u_T(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{50} e^{-n^2 c^2 t} \quad \ldots (7)
\]
At \( t = 0, \ u_T = 2x \quad \text{(from equation 3)}\)
\[
\Rightarrow \quad 2x = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{50}
\]
\[
\therefore \quad b_n = \frac{2}{50} \int_{0}^{50} 2x \sin \frac{n\pi x}{50} \, dx = \frac{2}{50} \left[ -\frac{50 x \cos \frac{n\pi x}{50} + 2500 \sin \frac{n\pi x}{50}}{n\pi \sin \frac{n\pi x}{50}} \right]_{0}^{50}
\]
\[
\quad = \frac{200}{n\pi} (-1)^{n+1}
\]
Putting the value of \( b_n \) in equation (7), we get
\[
u_T(x, t) = \sum_{n=1}^{\infty} \frac{200}{n\pi} (-1)^{n+1} \sin \frac{n\pi x}{50} e^{-n^2 c^2 t} \quad \ldots (8)
\]
Hence from (4) and (8), we get
\[
u(x, t) = \frac{200}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \sin \frac{n\pi x}{50} e^{-n^2 c^2 t} \quad \text{**Ans.**}\]
Example 21: A rod of length $l$ with insulated sides is initially at a uniform temperature $u$. Its ends are suddenly cooled 0ºC and are kept that temperature. Prove that the temperature function $u(x, t)$ is given by

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot e^{-\frac{n^2 \pi^2 \epsilon^2 t}{l^2}}$$

where

$$b_n = \frac{2}{l} \int_{0}^{l} u_0(x) \sin \frac{n\pi x}{l} \, dx.$$ 

Solution: Since the temperature distribution $u(x, t)$ is the solution of heat equation

$$\frac{\partial u}{\partial t} = \frac{\epsilon^2}{2} \frac{\partial^2 u}{\partial x^2} \quad \text{...(1)}$$

The solution of equation (1) is as follows

$$u(x, t) = (c_1 \cos px + c_2 \sin px) e^{-\frac{p^2 \epsilon^2 t}{2}} \quad \text{...(2)}$$

The boundary conditions are

(i) $u(0, t) = 0$ 
(ii) $u(l, t) = 0$

Making use of these boundary conditions the equation (2) takes the form

$$u(x, t) = c_2 c_3 \sin \frac{n\pi x}{l} e^{-\frac{n^2 \pi^2 \epsilon^2 t}{l^2}}$$

The general form of the solution is

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot e^{-\frac{n^2 \pi^2 \epsilon^2 t}{l^2}} \quad \text{...(3)}$$

and initial condition i.e., at $t = 0$, $u = u_0(x)$ (say)

$$u_0(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

which is half range Fourier sine series

$$b_n = \frac{2}{l} \int_{0}^{l} u_0(x) \sin \frac{n\pi x}{l} \, dx$$

Hence

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot e^{-\frac{n^2 \pi^2 \epsilon^2 t}{l^2}}$$

where

$$b_n = \frac{2}{l} \int_{0}^{l} u_0(x) \sin \frac{n\pi x}{l} \, dx. \quad \text{Proved.}$$
**Example 22:** Two ends $A$ and $B$ of a rod 20 cm long have the temperature at 30ºC and 80ºC respectively until steady state prevails. The temperature at the end are changed to 40ºC and 60ºC respectively find the temperature distribution in the rod.

**Solution:** The heat equation in one dimensional is
\[
\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}
\]  
...(1)

The boundary conditions are

(i) $u(0, t) = 30ºC$  
(ii) $u(20, t) = 80ºC$

In steady condition $\frac{\partial u}{\partial t} = 0$

\[\therefore \text{ From (1), we get } \frac{\partial^2 u}{\partial x^2} = 0, \text{ on integrating, we get } u(x) = c_1 x + c_2 \]

...\(2\)

at $x = 0, \quad u = 30, \quad \text{so } 30 = 0 + c_2 \quad \Rightarrow \quad c_2 = 30$

and at $x = 20, \quad u = 80 \quad \text{so } 80 = c_1 \times 20 + 30 \quad \Rightarrow \quad c_1 = \frac{5}{2}$

From equation (2), we get
\[u(x) = \frac{5x}{2} + 30\]

...(3)

Now the temperatures at $A$ and $B$ are suddenly changed we have again gain transient state.

If $u_t(x, t)$ is subsequent temperature function then the boundary conditions are

$u_t(0, t) = 40ºC$ and $u_t(20, t) = 60ºC$

and the initial condition i.e., at $t = 0$, is given by equation (3)

Since the subsequent steady state function $u_s(x)$ satisfies the equation

\[\frac{\partial^2 u_s}{\partial x^2} = 0 \quad \text{or} \quad \frac{d^2 u_s}{dx^2} = 0\]

The solution of above equation is
\[u_s(x) = c_3 x + c_4\]

...\(4\)

At $x = 0, \quad u_s = 40 \quad \Rightarrow \quad 40 = 0 + c_4 \quad \Rightarrow \quad c_4 = 40$

and at $x = 20, \quad u_s = 60 \quad \Rightarrow \quad 60 = 20 c_3 + 40 \quad \Rightarrow \quad c_3 = 1$

\[\therefore \text{ From (4), we get } u_s(x) = x + 40\]

...(5)

Thus the temperature distribution in the rod at time $t$ is given by

$u(x, t) = u_t(x, t) + u_s(x, t)$

\[\Rightarrow \quad u(x, t) = (x + 40) + u_t(x, t)\]

...(6)
where \( u_T(x, t) \) is the transient state function which satisfying the conditions

\[
\begin{align*}
    u_T(0, t) &= u_t(0, t) - u_s(0) = 40 - 40 = 0 \\
    u_T(20, t) &= u_t(20, t) - u_s(20) = 60 - 60 = 0 \\
    u_T(x, 0) &= u_t(x, 0) - u_s(x) = \frac{5x}{2} + 30 - x - 40 = \frac{3x}{2} - 10
\end{align*}
\]

and

\[
\begin{align*}
    u_T(0, t) &= u_t(0, t) - u_s(0) = 40 - 40 = 0 \\
    u_T(20, t) &= u_t(20, t) - u_s(20) = 60 - 60 = 0 \\
    u_T(x, 0) &= u_t(x, 0) - u_s(x) = \frac{5x}{2} + 30 - x - 40 = \frac{3x}{2} - 10
\end{align*}
\]

The general solution for \( u_T(x, t) \) is given by

\[
u_T(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20} e^{-\frac{n^2 \pi^2 c^2 t}{400}} \quad \text{...(7)}\]

At \( t = 0 \), from (7), we get

\[
\frac{3x}{2} - 10 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20}
\]

\[
\therefore \quad b_n = \frac{2}{20} \int_0^{20} \left( \frac{3x}{2} - 10 \right) \sin \frac{n\pi x}{20} dx
\]

\[
= \frac{1}{10} \left[ \left( \frac{3x}{2} - 10 \right) \left( -\frac{20}{n\pi} \cos \frac{n\pi x}{20} \right) \right]_0^{20} - \frac{3}{2} \left( -\frac{400}{n^2 \pi^2} \sin \frac{n\pi x}{20} \right) \right]_0^{20}
\]

\[
= \frac{1}{10} \left[ -20 \left( \frac{20}{n\pi} \right) (-1)^n - (-10) \left( \frac{20}{n\pi} \right) \right] = \frac{20}{n\pi} \left[ 2(-1)^n + 1 \right]
\]

Putting the value of \( b_n \) in equation (7), we get

\[
u_T(x, t) = -\frac{20}{n\pi} \sum_{n=1}^{\infty} \frac{2(-1)^n + 1}{n} \sin \frac{n\pi x}{20} e^{-\frac{n^2 \pi^2 c^2 t}{400}} \quad \text{...(8)}
\]

From (6) and (8), we get

\[
u(x, t) = (x + 40) - \frac{20}{n\pi} \sum_{n=1}^{\infty} \frac{2(-1)^n + 1}{n} \sin \frac{n\pi x}{20} e^{-\frac{n^2 \pi^2 c^2 t}{400}} \quad \text{Ans.} \]
1. Solve the boundary value problem \( \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \) with the conditions \( u(l, t) = 0 \) for all \( t \geq 0 \)
\[ \frac{\partial u}{\partial t}(0, t) = 0 \text{ and } u(x, 0) = 20x \text{ for } 0 < x < l. \]

\[
\text{Ans. } u(x, t) = \frac{40l}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \left( \frac{n\pi x}{l} \right) e^{-\frac{n^2 \pi^2 c^2 t}{l}}
\]

2. Solve \( \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \) for \( 0 < x < \pi, \ t > 0 \).
\[ u_x(0, t) = u_x(\pi, t) = 0 \text{ and } u(x, 0) = \sin x. \]

\[
\text{Ans. } u(x, t) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \cos 2nx \cdot e^{-4n^2 \pi^2 c^2 t}
\]

3. Solve \( \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \) with the conditions \( \frac{\partial u}{\partial t}(0, t) = 0, u(0, t) = 0, \ u(l, t) = 0 \) and \( u(x, 0) = lx - x^2, 0 \leq x \leq l \).

\[
\text{Ans. } u(x, t) = \frac{8l^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n - 1)^3} \sin \left( \frac{2n - 1}{2} \pi x \right) \cdot e^{-\frac{c^2 (2n-1)^2 \pi^2 t}{l}}
\]

4. Solve \( \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \) under the conditions (i) \( u \neq \infty \text{ if } t \to \infty \)
\[ (ii) \frac{\partial u}{\partial x} = 0 \text{ for } x = 0 \text{ and } x = l \text{ (iii) } u = lx - x^2 \text{ for } t = 0, 0 \leq x \leq l. \]

\[
\text{Ans. } u(x, t) = \frac{l^2}{6} - \frac{l^2}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2 \cos \left( \frac{2m\pi x}{l} \right)} e^{-\frac{4m^2 \pi^2 k t}{l}}
\]

5. The temperature distribution in a bar of length \( \pi \), which is perfectly insulated at the ends \( x = 0 \) and \( x = \pi \) is governed by the partial differential equation \( \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \). Assuming the initial temperature as \( u(x, 0) = \cos 2x \), find the temperature distribution at any instant of time.

\[
\text{Ans. } u(x, t) = e^{-4t} \cdot \cos 2x
\]

6. The ends \( A \) and \( B \) of a rod 30 cms long have their temperatures kept at 20ºC and 80ºC respectively until steady state conditions prevail. The temperature at the end \( B \) is then suddenly reduced to 60ºC and at the end \( A \) is raised to 40ºC and maintained so. Find \( u(x, t) \) at any time \( t \).

\[
\text{Ans. } u(x, t) = \frac{2x}{3} + 40 - \frac{40}{\pi} \sum_{n=1}^{\infty} \sin \left( \frac{n\pi x}{15} \right) e^{-\frac{n^2 \pi^2 225 t}{15}}
\]
7. A bar 40 cm long has originally a temperature of 0°C along all its length. At \( t = 0 \), the temperature at the end \( x = 0 \) is raised to 50°C while that at other end is raised to 100°C. Determine the resulting temperature function.

\[
\text{Ans. } u(x,t) = \frac{5x}{4} + 50 - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{50 + 100(-1)^n + 1}{n} \cdot \sin \frac{n\pi x}{40} \cdot e^{-\frac{\pi^2 c^2 t}{1600}}
\]

8. An insulated rod of length \( l \) has its ends \( A \) and \( B \) maintained at 0°C and 100°C respectively until steady state conditions prevail. If \( B \) is suddenly reduced to 0°C and maintained 0°C, find the temperature at a distance \( x \) from \( A \) at time \( t \).

\[
\text{Ans. } u(x,t) = \frac{200}{\pi} \sum_{n=1}^{\infty} \left( \frac{-1)^n}{n} \right) \sin \frac{n\pi x}{l} \cdot e^{-\frac{n^2 \pi^2 c^2 t}{l^2}}
\]

9. Solve \( \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \) under the conditions

(i) \( \frac{\partial u}{\partial x}(0, t) = 0 \),

(ii) \( \frac{\partial u}{\partial x}(5, t) = 0 \),

(iii) \( u(x, 0) = x \).

\[
\text{Ans. } u(x,t) = 5 - \frac{20}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2 \pi^2} \cdot e^{-\frac{(2n-1)^2 \pi^2 t}{5}}
\]

10. A rod of length \( l \) has its ends \( A \) and \( B \) kept 0°C and 120°C until steady state conditions prevail. If the temperature at \( B \) is reduced to 0°C and kept so while that of \( A \) is maintained find \( u(x, t) \).

\[
\text{Ans. } u(x,t) = \frac{240}{\pi} \sum_{n=1}^{\infty} \left( \frac{-1)^n}{n} \right) \sin \frac{n\pi x}{l} \cdot e^{-\frac{n^2 \pi^2 c^2 t}{l^2}}
\]

11. Solve \( \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \) under the boundary conditions

(i) \( u \) is finite when \( t \to \infty \),

(ii) \( u = 0 \) when \( x = 0 \) and \( x = \pi \) for all values of \( t \), and

(iii) \( u = x \) from \( x = 0 \) to \( x = \pi \) when \( t = 0 \).

\[
\text{Ans. } u(x, t) = 2 \sum_{n=1}^{\infty} \left( \frac{-1)^n}{n} \right) \sin nx \cdot e^{-n^2 \pi^2 c^2 t}
\]

5.6 TWO DIMENSIONAL HEAT EQUATION

We have

\[
\frac{\partial u}{\partial t} = c^2 (\nabla^2 u) \quad \text{...(i)}
\]

... In Section (5.5) (Alternate method)

where \( c^2 = \frac{k}{\rho} \), \( k \) is the thermal conductivity of the body, \( \rho \) is the specific heat of the material of the body and \( \rho \) is the density.

In case of two dimensional, we may suppose that \( z \)-coordinate is constant.

\[
\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 0 \quad \text{As } u(z) = \text{constant}
\]

\[
\frac{\partial^2 u}{\partial z^2} = 0
\]
\[ \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad \text{...(ii)} \]

From (i) and (ii), we get
\[ \frac{\partial u}{\partial t} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad \text{...(A)} \]

In steady state \( u \) always independent of \( t \) so that \( \frac{\partial u}{\partial t} = 0 \).
Hence from equation (A), we get
\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \quad \text{...(B)} \]

The equation (B) is known as Laplace’s equation.

### 5.6.1 Solution of Two Dimensional Heat Equation

We have
\[ \frac{\partial u}{\partial t} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad \text{...(i)} \]

Let \( u(x, y, t) = XYT \quad \text{...(ii)} \)
Putting the value of \( u(x, y, t) \) from (ii) in equation (i), we get
\[ XY \frac{dT}{dt} = Tc^2 \left( Y \frac{d^2 X}{dx^2} + X \frac{d^2 Y}{dy^2} \right) \]

\[ \Rightarrow \quad \frac{1}{C^2} \frac{dT}{dt} = \frac{1}{X} \frac{d^2 X}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} \]

There are three possibilities

(a) \[ \frac{1}{X} \frac{d^2 X}{dx^2} = 0, \quad \frac{1}{Y} \frac{d^2 Y}{dy^2} = 0, \quad \frac{1}{C^2} \frac{dT}{dt} = 0 \]

(b) \[ \frac{1}{X} \frac{d^2 X}{dx^2} = p_1^2, \quad \frac{1}{Y} \frac{d^2 Y}{dy^2} = p_2^2, \quad \frac{1}{C^2} \frac{dT}{dt} = p^2. \]

(c) \[ \frac{1}{X} \frac{d^2 X}{dx^2} = -p_1^2, \quad \frac{1}{Y} \frac{d^2 Y}{dy^2} = -p_2^2, \quad \frac{1}{C^2} \frac{dT}{dt} = -p^2. \]

where \( p^2 = p_1^2 + p_2^2 \).

Out of these three possibilities, we have to select that solution which suits the physical nature of the problem and the given boundary conditions.
Example 23: A thin rectangular plate whose surface is impervious to heat flow has \( t = 0 \) an arbitrary distribution of temperature \( f(x, y) \). Its four edges \( x = 0, x = a, y = 0 \) and \( y = b \) are kept at zero temperature. Determine the temperature at a point of the plate as \( t \) increases. (U.P.T.U. 2002)

**Solution:** The heat equation in two dimensional is

\[
\frac{\partial u}{\partial t} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad \text{...(1)}
\]

Let

\[ u = XYT \quad \text{...(2)} \]

From (1)

\[
XY \frac{dT}{dt} = Tc^2 \left( Y \frac{d^2X}{dx^2} + X \frac{d^2Y}{dy^2} \right)
\]

\[ \Rightarrow \quad \frac{1}{c^2T} \frac{dT}{dt} = \frac{1}{X} \frac{d^2X}{dx^2} + \frac{1}{Y} \frac{d^2Y}{dy^2} \quad \text{...(3)} \]

Since the physical nature of problem is periodic so we choosen the constant as follows:

From (3), we get

\[
\frac{1}{X} \frac{d^2X}{dx^2} = -p_1^2 \quad \Rightarrow \quad \frac{d^2X}{dx^2} + p_1^2 X = 0 \quad \Rightarrow \quad (D^2 + p_1^2) X = 0
\]

\[ \therefore \quad \text{The A.E. is} \quad m^2 + p_1^2 = 0 \quad \Rightarrow \quad m^2 = i^2 p_1^2 \quad \Rightarrow \quad m = \pm ip_1
\]

\[ \Rightarrow \quad X = c_1 \cos p_1 x + c_2 \sin p_1 x
\]

Again from (3), we get

\[
\frac{1}{Y} \frac{d^2Y}{dy^2} = -p_2^2 \quad \Rightarrow \quad (D^2 + p_2^2) Y = 0
\]

so

\[ Y = (c_3 \cos p_2 y + c_4 \sin p_2 y)
\]

and

\[ \frac{1}{c^2T} \frac{dT}{dt} = -p^2, \quad \text{where} \quad p^2 = p_1^2 + p_2^2.
\]

\[ \Rightarrow \quad \frac{dT}{T} = -p^2 c^2 dt \quad \Rightarrow \quad \log_e T = -p^2 c^2 t + \log_e c_5
\]

\[ \Rightarrow \quad T = c_5 e^{-p^2 c^2 t}
\]
Hence from equation (2), we get
\[ u = (c_1 \cos p_1 x + c_2 \sin p_1 x)(c_3 \cos p_2 y + c_4 \sin p_2 y)c_5 e^{-p^2 c^2 t} \] ... (4)

Now the boundary conditions are:
(i) \( u(0, y, t) = 0 \),
(ii) \( u(a, y, t) = 0 \),
(iii) \( u(x, 0, t) = 0 \); and (iv) \( u(x, b, t) = 0 \)

Using first boundary condition in (4), we get
\[ 0 = c_1 (c_3 \cos p_2 y + c_4 \sin p_2 y)c_5 e^{-p^2 c^2 t} \]
\[ \Rightarrow c_1 = 0 \quad \text{(otherwise } u(x, y, t) = 0) \]

From (4), we get
\[ u = c_2 c_3 \sin p_1 x (c_3 \cos p_2 y + c_4 \sin p_2 y)e^{-p^2 c^2 t} \] ... (5)

Using second boundary condition in (5), we get
\[ 0 = c_2 c_3 \sin p_1 a (c_3 \cos p_2 y + c_4 \sin p_2 y)e^{-p^2 c^2 t} \]
\[ \Rightarrow \sin p_1 a = 0 = \sin m\pi \quad \Rightarrow \quad p_1 = \frac{m\pi}{a} \]

Next using 3rd boundary condition in (5), we get
\[ 0 = c_2 c_3 c_5 \sin p_1 x \cdot e^{-p^2 c^2 t} \quad \Rightarrow \quad c_3 = 0 \quad \text{(otherwise } u(x, y, t) = 0) \]

From (5), we get
\[ u = c_2 c_4 c_5 \sin p_1 x \cdot e^{-p^2 c^2 t} \] ... (6)

And using 4th boundary condition in (6), we get
\[ 0 = c_2 c_4 c_5 \sin p_1 x \cdot e^{-p^2 c^2 t} \]
\[ \Rightarrow \quad \sin p_2 b = 0 = \sin n\pi \quad \Rightarrow \quad p_2 = \frac{n\pi}{b} \]

Since
\[ p^2 = p_1^2 + p_2^2 \]
so
\[ p^2 = \pi^2 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \]

Putting the values of \( p_1 \), \( p_2 \) and \( p \) in equation (6), we get
\[ u(x, y, t) = c_2 c_4 c_5 \sin \frac{m\pi x}{a} \cdot \sin \frac{n\pi y}{b} e^{-\pi^2 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right) t} \]

The general form of solution is
\[ u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \cdot \sin \frac{n\pi y}{b} e^{-\pi^2 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right) t} \] ... (7)
and the initial condition is

\[ u(x, y, 0) = f(x, y) \]

From (7), we get

\[ f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \]

which is the double Fourier series of \( f(x, y) \).

\[ A_{mn} = \frac{2}{a} \frac{2}{b} \int_{x=0}^{a} \int_{y=0}^{b} f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx \, dy \]

Hence the equation (7) is required temperature distribution with the equation (8).  Ans.

**Example 24:** Solve the following Laplace equation:

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \]

in a rectangle with \( u(0, y) = 0, u(a, y) = 0, u(x, 0) = 0 \) and \( u(x, 0) = f(x) \) along x-axis

[U.P.T.U. 2008]

**Solution:** Let \( u(x, y) = X(x)Y(y) \).

\[ \frac{\partial^2 u}{\partial x^2} = X \frac{d^2 Y}{d y^2}, \quad \frac{\partial^2 u}{\partial y^2} = Y \frac{d^2 X}{d x^2} \]

Putting these values in given equation, we get

\[ Y \frac{d^2 X}{d x^2} + X \frac{d^2 Y}{d y^2} = 0 \]

\[ \Rightarrow \quad Y \frac{d^2 X}{d x^2} = -X \frac{d^2 Y}{d y^2} \Rightarrow \quad \frac{d^2 X}{d x^2} = \frac{-1}{X} \frac{d^2 Y}{d y^2} = -p^2 \quad \text{(say)} \]

Taking first two terms, we get

\[ \frac{d^2 X}{d x^2} + p^2 X = 0 \]

\[ \Rightarrow \quad X = c_1 \cos px + c_2 \sin px \]

And taking last two terms, we get

\[ \frac{d^2 Y}{d y^2} - p^2 Y = 0 \]

\[ \Rightarrow \quad Y = c_3 e^{py} + c_4 e^{-py} \]

From (i)

\[ u(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \]

Putting \( x = 0 \), we get

\[ 0 = c_1(c_3 e^{py} + c_4 e^{-py}) = 0 \quad \Rightarrow \quad c_1 = 0 \quad \text{otherwise} \]

\[ u = 0 \]
APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

From (iii) \[ u(x, y) = c_2 \sin px (c_3 e^{py} + c_4 e^{-py}) \] ...(iv)

Putting \( x = a \), in (iv), we get

\[ 0 = c_2 \sin pa (c_3 e^{py} + c_4 e^{-py}) \Rightarrow \sin pa = 0 \]

or

\[ \sin pa = \sin n\pi \Rightarrow p = \frac{n\pi}{a} \]

From (iv), \[ u(x, y) = c_2 \sin \frac{n\pi x}{a} (c_3 e^{a-y} + c_4 e^{-a+y}) \] ...(v)

Putting \( y = b \) in (v), we get

\[ 0 = c_2 \sin \frac{n\pi x}{a} \left( \frac{n\pi}{a} e^{a-y} + \frac{n\pi}{a} e^{-a+y} \right) \]

\[ \Rightarrow c_3 e^{a-y} + c_4 e^{-a+y} = 0 \Rightarrow c_4 = -c_3 e^{\frac{n\pi b}{a}} \] ...(ii)

From (v), we get

\[ u(x, y) = c_2 c_3 \cdot \sin \frac{n\pi x}{a} \left( \frac{n\pi}{e^a} - e^{-\frac{n\pi}{a}(y-b)} \right) \]

or \[ u(x, y) = \sum_{n=1}^{\infty} b_n \cdot \sin \frac{n\pi x}{a} \left( \frac{n\pi}{e^a} - e^{-\frac{n\pi}{a}(y-b)} \right) \] ...(vi)

Putting \( y = 0 \), we get

\[ f(x) = \sum_{n=1}^{\infty} b_n \cdot \sin \frac{n\pi x}{a} \left( 1 - e^{-\frac{2\pi b}{a}} \right) \] ...(vii)

which is Fourier sine series

\[ b_n \left( 1 - e^{-\frac{2\pi b}{a}} \right) = \frac{2}{a} \int_0^a f(x) \cdot \sin \frac{n\pi x}{a} \, dx \]

\[ \Rightarrow b_n = \frac{2}{a \left( 1 - e^{-\frac{2\pi b}{a}} \right)} \int_0^a f(x) \cdot \sin \frac{n\pi x}{a} \, dx \] ...(viii)

Hence equation (vi) is required solution with coefficient \( b_n \) (equations (viii)).

Example 25: An infinitely long uniform plate is bounded by two parallel edges and an end at right angles to them. The breadth is \( \pi \), the end is maintained at a temperature 100ºC at all points and other edges are at 0ºC. Show that the steady state temperature is given by

\[ u(x, y) = \frac{400}{\pi} \left[ e^{-y} \sin x + \frac{1}{3} e^{-3y} \sin 3x + \frac{1}{5} e^{-5y} \sin 5x + \cdots \right] \]
Solution: The steady state temperature \( u(x, y) \) is the solution of

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \ldots(1)
\]

The boundary conditions are

(i) \( u(0, y) = 0 = u(\pi, y) \) for all \( y \geq 0 \)

(ii) \( u(x, \infty) = \lim_{y \to \infty} (x, y) = 0 \) for \( 0 \leq x \leq \pi \)

and initial condition is (iii) \( u(x, 0) = 100 \)

Let \( u(x, y) = XY \) \ldots(2)

Putting the value of \( u \) in equation (1), we get

\[
Y \frac{d^2X}{dx^2} + X \frac{d^2Y}{dy^2} = 0
\]

\[\Rightarrow \quad Y \frac{d^2X}{dx^2} = -X \frac{d^2Y}{dy^2} \quad \Rightarrow \quad \frac{1}{X} \frac{d^2X}{dx^2} = -\frac{1}{Y} \frac{d^2Y}{dy^2} = -p^2 \quad \text{(say)} \quad \ldots(3)
\]

Taking first two member, we get

\[
\frac{d^2X}{dx^2} + p^2 X = 0
\]

\[\Rightarrow \quad X = c_1 \cos px + c_2 \sin px \]

And now taking last two member of equation (3), we get

\[
\frac{d^2Y}{dy^2} - p^2 Y = 0 \quad \Rightarrow \quad Y = c_3 e^{py} + c_4 e^{-py}
\]

Putting the values of \( X \) and \( Y \) in equation (2), we get

\[
u(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \quad \ldots(4)
\]

At \( x = 0, \ u = 0 \)

\[0 = c_1 (c_3 + c_4) \quad \Rightarrow \quad c_1 = 0.
\]

From equation (4), we have

\[
u(x, y) = c_2 \sin px (c_3 e^{py} + c_4 e^{-py}) \quad \ldots(5)
\]

Again at \( x = \pi, \ u = 0 \)

\[0 = c_2 \sin p\pi (c_3 + c_4)\]

\[\Rightarrow \quad \sin p\pi = 0 = \sin n\pi \quad \Rightarrow \quad p = \frac{n\pi}{\pi} = n \quad \text{otherwise} \quad u(x, y) = 0\]
Putting the value of $p$ in equation (5), we get

$$u(x, y) = c_2 \sin nx(c_3 e^{ny} + c_4 e^{-ny})$$

...(6)

Now using (ii) boundary condition in equation (6) i.e., when $y \to \infty$, $u = 0$, we get

$$0 = c_2 \sin nx \lim_{y \to \infty} (c_3 e^{ny} + c_4 e^{-ny})$$

$$0 = c_2 \sin nx \left\{ c_3 \lim_{y \to \infty} e^{ny} + 0 \right\}$$

As $\lim_{y \to \infty} e^{-ny} \to 0$

$$\Rightarrow \quad c_3 = 0$$

(Otherwise $u(x, y) = 0$)

Putting the value of $c_3$ in equation (6), we get

$$u(x, y) = c_2 c_4 \sin nx e^{-ny}$$

∴ The general form of solution may be written as

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin nx e^{-ny}$$

...(7)

Using (iii) condition (i.e., at $y = 0$) in equation (7), we get

$$100 = \sum_{n=1}^{\infty} b_n \sin nx$$

which is Fourier half range sine series in $0 \leq x \leq \pi$

∴

$$b_n = \frac{2}{\pi} \int_{0}^{\pi} 100 \cdot \sin nx dx = \frac{200}{n \pi} \left[ -\cos nx \right]_{0}^{\pi} = \frac{200}{n \pi} \left[ -\cos n\pi + \cos 0 \right]$$

$$b_n = \frac{200}{n \pi} \left[ (-1)^n + 1 \right] = \begin{cases} \frac{400}{n \pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Hence from equation (7), we get

$$u(x, y) = \frac{400}{\pi} \left[ e^{-y} \sin x + \frac{1}{3} e^{-3y} \sin 3x + \frac{1}{5} e^{-5y} \sin 5x + \cdots \right]. \quad \text{Proved.}$$

Example 26: Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ which satisfies the conditions

$$u(0, y) = u(l, y) = u(x, 0) = 0 \quad \text{and} \quad u(x, l) = \sin \left( \frac{n \pi x}{l} \right). \quad \text{(U.P.T.U. 2004)}$$

Solution: We have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

...(1)
The solution of equation (1) can be obtained as in (Example 26, equation (4) on page no. 494) is given below:

\[ u(x, y) = (c_1 \cos px + c_2 \sin px) \left( c_3 e^{py} + c_4 e^{-py} \right) \]  
\[ \ldots (2) \]

At 
\[ x = 0, \quad u = 0 \]
\[ 0 = c_1(c_3 e^{py} + c_4 e^{-py}) \Rightarrow c_1 = 0 \quad \text{[otherwise } u(x, y) = 0 \text{]} \]

Putting the value of \( c_1 \) in equation (2), we get

\[ u(x, y) = c_2 \sin px(c_3 e^{py} + c_4 e^{-py}) \]  
\[ \ldots (3) \]

at \( x = l, \quad u = 0 \)
\[ 0 = c_2 \sin pl(c_3 e^{py} + c_4 e^{-py}) \]
\[ \Rightarrow \quad \sin pl = 0 = \sin np \quad \Rightarrow \quad p = \frac{n\pi}{l}. \]

From equation (3), we get

\[ u(x, y) = c_2 \sin \frac{n\pi x}{l} \left( c_3 e^{\frac{n\pi y}{l}} + c_4 e^{-\frac{n\pi y}{l}} \right) \]  
\[ \ldots (4) \]

At 
\[ y = 0, \quad u = 0, \quad \text{we get} \]
\[ 0 = 2c_2 \sin \frac{n\pi x}{l}(c_3 + c_4) \Rightarrow c_3 + c_4 = 0 \Rightarrow c_4 = -c_3 \]

Putting the value of \( c_4 \) in equation (4), we get

\[ u(x, y) = c_2 c_3 \sin \frac{n\pi x}{l} \left( e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}} \right) = 2c_2 c_3 \sin \frac{n\pi x}{l} \sinh \frac{n\pi y}{l} \sin\theta = \frac{e^\theta - e^{-\theta}}{2} \]

The general form of above solution is given as

\[ u(x, y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \sinh \frac{n\pi y}{l} \quad (2c_1 c_2 = b_n) \]  
\[ \ldots (5) \]

Putting \( y = a \) and \( u = \sin \frac{n\pi x}{l} \) in equation (5), we get

\[ \sin \frac{n\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \sinh \frac{n\pi a}{l} \]

Equating the coefficient of \( \sin \frac{n\pi x}{l} \) on both sides, we get

\[ 1 = b_n \sinh \frac{n\pi a}{l} \Rightarrow b_n = \frac{1}{\sin h \frac{n\pi a}{l}} \]
and

\[ b_1 = b_2 = b_3 = \ldots \ldots = b_{n-1} = 0 \]

Hence, from equation (5), we get

\[ u(x, y) = \frac{\sin(n\pi y/l)}{\sin(n\pi a/l)} \sin \frac{n\pi y}{l} \sinh \frac{n\pi a}{l} = 0. \]

**Ans.**

**Example 27:** A thin rectangular plate whose surface is impervious to heat flow has \( t = 0 \) an arbitrary distribution of temperature \( f(x, y) \). Its four edges \( x = 0, x = a, y = 0, y = b \) are kept at 0°C temperature. Determine the temperature at a point of a plate as \( t \) increases. Discuss when

\[ f(x, y) = \beta \sin \left( \frac{\pi x}{a} \right) \sin \left( \frac{\pi y}{b} \right). \]

**Solution:** Two dimensional heat flow equation is

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2} \frac{\partial u}{\partial t} \]  

\[ \ldots (1) \]

Boundary conditions are

\[ u(0, y, t) = 0 \]
\[ u(a, y, t) = 0 \]
\[ u(x, 0, t) = 0 \]
\[ u(x, b, t) = 0 \]

and the initial condition is

\[ u(x, y, t) = f(x, y) \text{ at } t = 0. \]

The solution of equation (1) is given below (Example 23, equation (4) on page 489)

\[ u = (c_1 \cos p_1 x + c_2 \sin p_1 x)(c_3 \cos p_2 y + c_4 \sin p_2 y)c_5e^{-\gamma^2 t} \]  

\[ \ldots (2) \]

At

\[ x = 0, \quad u = 0, \text{ then we get} \]

\[ u(0, y, t) = 0 = c_1(c_3 \cos p_2 y + c_4 \sin p_2 y)c_5e^{-\gamma^2 t}; \]

\[ \Rightarrow \quad c_1 = 0 \]

\[ \therefore \text{ from } (2), \quad u(x, y, t) = c_2 c_5 \sin p_1 x(c_3 \cos p_2 y + c_4 \sin p_2 y)(e^{-\gamma^2 t}) \]  

\[ \ldots (3) \]

from (3), \[ u(a, y, t) = 0 = c_2 c_3 \sin p_1 a(c_3 \cos p_2 y + c_4 \sin p_2 y)e^{-\gamma^2 t}; \]

\[ \Rightarrow \quad \sin p_1 a = 0 = \sin m\pi (m \in I) \]

\[ \therefore \quad p_1 = \frac{m\pi}{a}. \]

From (3), \[ u(x, y, t) = c_2 c_3 \sin \frac{m\pi x}{a}(c_3 \cos p_2 y + c_4 \sin p_2 y)(e^{-\gamma^2 t}) \]  

\[ \ldots (4) \]
\[ u(x, 0, t) = 0 = c_2 c_5 \sin \frac{m \pi x}{a} \cdot c_3 e^{-c^2 \pi^2 t} \]

\[ \Rightarrow \quad c_5 = 0. \]

\[ \therefore \text{from (4),} \quad u(x, y, t) = c_2 c_5 c_4 \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} e^{-c^2 \pi^2 t} \quad \cdots (5) \]

\[ u(x, b, t) = 0 = c_2 c_5 c_4 \sin \frac{m \pi x}{a} \sin \frac{n \pi b}{b} e^{-c^2 \pi^2 t} \]

\[ \Rightarrow \quad \sin \frac{p_2 b}{b} = 0 = \sin n \pi (n \in I) \]

\[ p_2 b = n \pi \]

\[ \Rightarrow \quad p_2 = \frac{n \pi}{b} \]

\[ \therefore \text{from (5),} \quad u(x, y, t) = c_2 c_5 c_4 \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} e^{-c^2 \pi^2 t} \]

\[ = A_{mn} \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} e^{-c^2 \pi^2 t} \quad \cdots (6) \]

where \( c_2 c_5 c_4 = A_{mn} \)

But,

\[ p^2 = p_1^2 + p_2^2 = \frac{m^2 \pi^2}{a^2} + \frac{n^2 \pi^2}{b^2} \]

or

\[ p_{mn}^2 = \pi^2 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \]

By using \( p_{mn}^2 \), equation (6) becomes,

\[ u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} e^{-c^2 \pi^2 t} \quad \cdots (7) \]

which is the most general solution.

\[ \therefore \quad u(x, y, 0) = f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} \]

which is the double Fourier half range sine series for \( f(x, y) \).

\[ A_{mn} = \frac{2}{a} \frac{2}{b} \int_{x=0}^{a} \int_{y=0}^{b} \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} f(x, y) dx dy \quad \text{Ans.} \]

when

\[ f(x, y) = \beta \sin \left( \frac{\pi x}{a} \right) \sin \left( \frac{\pi y}{b} \right) \]

\[ : \quad A_{mn} = \frac{2}{a} \frac{2}{b} \int_{x=0}^{a} \int_{y=0}^{b} \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} \beta \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} dx dy \]
\[ v(x, y, t) = 0 \] when \( f(x, y) = \beta \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \) (proved.)

**Example 28:** The edges of a thin plate of side \( \pi \) cm are kept at temperature zero and faces insulated. The initial temperature is \( u(x, y, 0) = f(x, y) = xy(\pi - x)(\pi - y) \). Determine temperature in the plate at time \( t \).

**Solution:** Referred to solution of Example 4, where, \( a = \pi \) and \( b = \pi \) and \( f(x, y) = xy(\pi - x)(\pi - y) \).

\[
p_{mn}^2 = \pi^2 \left( \frac{m^2}{\pi^2} + \frac{n^2}{\pi^2} \right) = m^2 + n^2
\]

\[
u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin mx \sin ny e^{-c^2(m^2+n^2)t} \quad \text{...}(1)
\]

\[
u(x, y, 0) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin mx \sin ny \quad \text{...}(2)
\]

\[
A_{mn} = \frac{4}{\pi^2} \int_0^\pi \int_0^\pi xy(x - \hat{x})(\pi - y) \sin mx \sin ny dx dy
\]

\[
= \frac{4}{\pi^2} \int_0^\pi (\pi x - x^2) \sin mx dx \int_0^\pi (\pi y - y^2) \sin ny dy \quad \text{...}(3)
\]
Now \[ \int_{0}^{\pi} (\pi x - x^2) \sin mx dx = \left[ (\pi x - x^2) \left( -\cos mx \right) \right]_{0}^{\pi} \]
\[= \frac{2}{m^3} [1 - (-1)^m] \]
and similarly
\[\int_{0}^{\pi} (\pi y - y^2) \sin ny dy = \frac{2}{n^3} [1 - (-1)^n] \]

\[\therefore A_{mn} = \frac{16}{\pi^2 m n} [1 - (-1)^m][1 - (-1)^n] \]

or
\[A_{mn} = \begin{cases} 0, & \text{when } m = 2p \text{ and } n = 2q \\ \frac{64}{\pi^2 (2p-1)^3 (2q-1)^3}, & \text{when } m = 2p-1 \text{ and } n = 2q-1 \end{cases} \]

Hence from (2)
\[u(x, y, t) = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} A_{pq} \sin(2p-1)x \sin(2q-1)y e^{-\mu_{pq}^2 t} \quad \text{Ans.} \]

where
\[\mu_{pq}^2 = c^2 \left[ (2p-1)^2 + (2q-1)^2 \right] \]

and
\[A_{pq} = \frac{64}{\pi^2 (2p-1)^3 (2q-1)^3}. \]

**Example 29:** A square plate is bounded by the lines \(x = 0, y = 0, x = 10 \) and \(y = 10\). Its faces are insulated. The temperature along the upper horizontal edge is given by \(u(x, 10) = x(10 - x)\), while the other three edges are kept at 0ºC. Find the steady state temperature in the plate.

**Solution:** The steady state temperature \(u(x, y)\) is the solution of the equation
\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{(1)}
\]
The solution of equation (1) is given below
\[u(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \quad \text{(2)} \]
[See eqn. (4) on page no. 491]
The boundary conditions are
\[
\begin{align*}
&u(0, y) = u(10, y) = 0, & 0 \leq y \leq 10 \\
u(x, 0) = 0, & 0 \leq x \leq 10 \\
u(x, 10) = x(10 - x); & 0 \leq x \leq 10
\end{align*}
\]
and
\[u(x, 10) = x(10 - x); \quad 0 \leq x \leq 10 \quad \text{(A)} \]
APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

Putting \( x = 10 \) and \( u = 0 \) in (2), we get
\[
0 = (c_1 e^{py} + c_4 e^{-py}) \quad \Rightarrow \quad c_1 = 0
\]
.: From (2), we have
\[
u(x, y) = c_2 \sin px(e^{py} + c_4 e^{-py}) \quad \text{...(3)}
\]
Next putting \( x = 10 \) in (3), we get
\[
0 = c_2 \sin 10 p(e^{py} + c_4 e^{-py}) \quad \Rightarrow \quad \sin 10 p = 0 = \sin n\pi \quad \Rightarrow \quad p = \frac{n\pi}{10}.
\]
Again putting \( y = 0 \) and \( u = 0 \) in equation (3), we get
\[
0 = c_2 \sin px(c_3 + c_4) \quad \Rightarrow \quad c_3 + c_4 = 0 \quad \Rightarrow \quad c_4 = -c_3
\]
Putting the values of \( c_4 \) and \( p \) in equation (3), we get
\[
u(x, y) = c_2 c_3 \sin \frac{n\pi x}{10} \left( e^{10} - e^{-\frac{n\pi x}{10}} \right) = 2c_2 c_3 \sin \frac{n\pi x}{10} \sin h \frac{n\pi y}{10}
\]
The general solution is given by
\[
u(x, y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{10} \cdot \sin h \frac{n\pi y}{10} \quad \text{...(4)}
\]
Putting \( y = 10 \) and \( u = x(10 - x) \) in equation (4), we get
\[
x(10 - x) = \sum_{n=1}^{10} b_n \sin \frac{n\pi x}{10} \cdot \sin h n\pi
\]
.: \( b_n \sin h n\pi = \frac{2}{10} \int_{0}^{10} x(10 - x) \sin \frac{n\pi x}{10} \quad | \sin h n\pi \text{ is a constant}
\]
\[
= \frac{1}{5} \left[ -\frac{1000}{n^3 \pi^3} \cos n\pi + \frac{1000}{n^3 \pi^3} \right]_{0}^{10}
\]
\[
= \frac{1}{5} \left[ -2 \times \frac{1000}{n^3 \pi^3} \cos n\pi + 2 \times \frac{1000}{n^3 \pi^3} \right]
\]
\[
= \frac{2000}{5n^3 \pi^3} [(-1)^n + 1] = \begin{cases} \frac{800}{n^3 \pi^3}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}
\]
.: \( b_n = \begin{cases} \frac{800}{n^3 \pi^3 \sin h n\pi}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases} \)
Putting the value of \( b_n \) in equation (4), we get
\[
\begin{align*}
    u(x, y) &= \frac{800}{\pi^3} \sum_{m=1}^{\infty} \frac{\sin((2m-1)\pi x)}{(2m-1)^3 \sin h((2m-1)\pi y)}; \quad \text{where } n = 2m - 1. \quad \text{Ans.}
\end{align*}
\]

**Example 30:** A rectangular metal plate is bounded by \( x = 0, x = a, y = 0 \) and \( y = b \). The three sides \( x = 0, x = a \) and \( y = b \) are insulated and the edge \( y = 0 \) is kept at \( u_0 \cos \frac{\pi x}{a} \). Show that the temperature in the steady state is
\[
    u(x, y) = u_0 \sec h \left( \frac{(b-a)\pi}{a} \right) \cos h \left( \frac{\pi(b-y)}{a} \right) \cos \left( \frac{\pi x}{a} \right) 
\]

**Solution:** The steady state temperature \( u(x, y) \) is the solution of the equation
\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \ldots(1)
\]
Since sides \( x = 0, x = a \) and \( y = b \) are insulated
\[
\frac{\partial u}{\partial x} \bigg|_{x=0} = 0 \quad \ldots(2)
\]
\[
\frac{\partial u}{\partial x} \bigg|_{x=a} = 0 \quad \ldots(3)
\]
\[
\frac{\partial u}{\partial x} \bigg|_{y=b} = 0 \quad \ldots(4)
\]
Also
\[
    u(x, a) = u_0 \cos \left( \frac{\pi x}{a} \right) \quad \ldots(5)
\]
The solution of (1) is
\[
    u(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \quad \ldots(6)
\]
Now
\[
\frac{\partial u}{\partial x} = (-c_1 p \sin px + c_2 p \cos px)(c_3 e^{py} + c_4 e^{-py})
\]
\[
\frac{\partial u}{\partial x} \bigg|_{x=0} = 0 \quad \Rightarrow \quad c_2 p (c_3 e^{py} + c_4 e^{-py}) = 0
\]
\[
\Rightarrow \quad c_2 = 0 \quad \text{(:} \quad c_3 e^{py} + c_4 e^{-py} \neq 0) \]
\[
\frac{\partial u}{\partial x} \bigg|_{x=a} = 0 \quad \Rightarrow \quad -c_1 p \sin p(a(c_3 e^{py} + c_4 e^{-py})) = 0
\]
\[ \Rightarrow \quad \sin p a = 0 \quad (\because c_1 \neq 0 \text{ otherwise } u(x, y) \text{ would be a trivial solution}) \]

\[ \Rightarrow \quad p a = n p \quad (n \text{ is an integer}) \]

\[ \Rightarrow \quad p = \frac{n \pi}{a} \]

Now

\[ \frac{\partial u}{\partial y} = c_1 \cos px(c_3 e^{py} - c_4 e^{-py}). \]

\[ \left( \frac{\partial u}{\partial y} \right)_{y=b} = 0 \Rightarrow c_1 \cos px(c_3 e^{pb} - c_4 e^{-pb}) = 0 \]

\[ \Rightarrow \quad c_3 e^{pb} - c_4 e^{-pb} = 0 \]

\[ \Rightarrow \quad p(c_3 e^{pb} - c_4 e^{-pb}) = 0 \]

\[ \Rightarrow \quad c_3 e^{pb} = c_4 e^{-pb} \]

\[ \Rightarrow \quad c_4 = c_3 e^{2pb} \]

\[ \therefore \text{From (6)} \quad u(x, y) = c_1 \cos px(c_3 e^{py} + c_3 e^{2pb} e^{-py}) \quad \text{where} \quad p = \frac{n \pi}{a} \]

\[ = c_3 c_1 \cos px \left( e^{py} + \frac{e^{pb} e^{-py}}{e^{-pb}} \right) \]

\[ = \frac{c_1 c_3}{e^{-pb}} \cos px \left( e^{py} e^{pb} + e^{pb} e^{-py} \right) \]

\[ = c_3 c_1 e^{pb} \cos px \left[ e^{-p(b-y)p} + e^{(b-y)p} \right] \]

\[ = 2 c_1 c_3 e^{pb} \cos px \cos h(b-y)p \]

\[ = K \cos \left( \frac{n \pi x}{a} \right) \cos h \left( \frac{(b-y)n \pi}{a} \right) \quad \text{where} \quad K = 2 c_1 c_3 e^{pb} \]

\[ \therefore \text{The most general solution of (1)} \]

\[ u(x, y) = \sum_{n=1}^{\infty} A_n \cos \left( \frac{n \pi x}{a} \right) \cos h \left( \frac{(b-y)n \pi}{a} \right) \]

where \( A_n \)'s \( a \) are to be determined.

Since \( u(x, a) = u_0 \cos \left( \frac{\pi x}{a} \right) \) for \( 0 < x < a \),
\[
\begin{align*}
\text{u}(x, a) &= \sum_{n=1}^{\infty} A_n \cos \left( \frac{n\pi x}{a} \right) \cos h \left( \frac{(b-a)n\pi}{a} \right) = u_0 \cos \left( \frac{\pi x}{a} \right) \\
\therefore \quad u_0 \cos \left( \frac{\pi x}{a} \right) &= \sum_{n=1}^{\infty} A_n \cos \left( \frac{n\pi x}{a} \right) \cos h \left( \frac{(b-a)n\pi}{a} \right) \\
&= A_1 \cos \left( \frac{\pi x}{a} \right) \cosh \left( \frac{(b-a)\pi}{a} \right) + A_2 \cos \left( \frac{2\pi x}{a} \right) \cosh \left( \frac{(b-a)2\pi}{a} \right) + \cdots
\end{align*}
\]

Comparing both sides, we get

\[
A_1 = u_0 \sec h \left( \frac{(b-a)\pi}{a} \right)
\]

and

\[
A_2 = A_3 = \cdots = 0
\]

\[
\therefore \quad u(x, y) = u_0 \sec h \left( \frac{(b-a)\pi}{a} \right) \cos \left( \frac{\pi x}{a} \right) \cosh \left( \frac{(b-a)\pi}{a} \right) \cos \left( \frac{\pi y}{a} \right)
\]

or

\[
u(x, y) = u_0 \sec h \left( \frac{(b-a)\pi}{a} \right) \cos \left( \frac{(b-a)\pi}{a} \right) \cos \left( \frac{\pi x}{a} \right) \cosh \left( \frac{(b-a)2\pi}{a} \right) + \cdots
\]

Proved.

**Exercise 5.3**

1. An infinite long plane uniform plate is bounded by two parallel edges and an end at right angles to them. The breadth is \( \pi \). This end is maintained at temperature \( u_0 \) at all points and other are at zero temperature. Determine the temperature at any point of the plate in the steady state. 

   \[ \text{Ans. } u(x, y) = \frac{4u_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin(2n-1)xe^{-(2n-1)y} \]

2. A square plate is bounded by the lines \( x = 0, y = 0, x = 20 \) and \( y = 20 \). Its faces are insulated. The temperature along the upper horizontal edge is given by \( u(x, 20) = x(20 - x) \) when \( 0 < x < 20 \), while the other three edges are kept at 0°C. Find the steady temperature in the rod.

   \[ \text{Ans. } u(x, y) = \frac{3200}{\pi^3} \sum_{m=1}^{\infty} \frac{\sin^2 \left( \frac{2m-1}{20} \right) \sinh \left( \frac{2m-1}{20} \pi y \right)}{\sin A(2m-1)\pi}; 2m-1 = n \]

3. A rectangular plate with insulated surfaces is 10 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along the short edge \( y = 0 \) is given by

   \[
u(x, 0) = 20x, \quad 0 < x \leq 5
   = 20(10 - x), \quad 5 < x < 10.
\]
while the two long edges \( x = 0 \) and \( x = 10 \) as well as other short edges are kept at 0ºC. Find the steady state temperature at any point \((x, y)\) of the plate.

\[
\text{Ans. } u(x, y) = \frac{800}{\pi^2} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(2m-1)^2} \sin \left( \frac{(2m-1)\pi x}{10} \right) e^{-\frac{(2m-1)\pi y}{10}}
\]

4. A long rectangular plate of width \( l \) with insulated surfaces has its temperature \( u \) equal to zero on both the long sides and one of the shorter sides so that \( u(0, y) = u(l, y) = u(x, 0) = kx \). Find \( u(x, y) \).

\[
\text{Ans. } u(x, y) = \frac{2kl}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \left( \frac{n\pi x}{l} \right) e^{-\frac{n\pi y}{l}}
\]

5. A long rectangular plate has its surface insulated and the two long sides as well one of the short sides are kept at 0ºC while the other sides is kept at \( u(x, 0) = 3x \) and the length being 5 cm. Find \( u(x, y) \).

\[
\text{Ans. } u(x, y) = \frac{30}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \left( \frac{n\pi x}{5} \right) e^{-\frac{n\pi y}{5}}
\]

6. A square plate has its faces and the edge \( y = 0 \) insulated. Its edges \( x = 0 \) and \( x = \pi \) are kept at 0ºC and the edge \( y = \pi \) is kept at temperature \( f(x) \). Find the steady state temperature.

\[
\text{Ans. } u(x, y) = \sum_{n=1}^{\infty} b_n \frac{\sin nx \cosh n\pi}{\cosh n\pi}, \text{ where } b_n = \frac{2}{\pi} \int_0^\pi \sin nx f(x) dx
\]

7. A long rectangular plate has its surface insulated and the two long sides as well as one of the short sides are maintained at 0ºC. Find an expression for the steady state temperature \( u(x, y) \) if the short side \( y = 0 \) is 30 cm long and kept at 40ºC.

\[
\text{Ans. } u(x, y) = \frac{160}{\pi} \sum_{m=1}^{\infty} \frac{1}{2m-1} \sin \left( \frac{(2m-1)\pi x}{30} \right) e^{-\frac{(2m-1)\pi y}{30}}
\]

5.6.2 Two Dimensional Heat Flow in Polar Coordinate

Let \( c_1 \) and \( c_2 \) be two arcs of circles of radius \( r \) and \( r + \delta r \) respectively. Here we consider an element between the circles \( c_1 \) and \( c_2 \) and the lines \( OP \) and \( OQ \) i.e., \((\theta \text{ and } \theta + \delta\theta)\) of any sheet of conducting material of uniform density \( \rho \).

We know that the area of portion \( PP'Q'O = \delta\theta \delta\theta \)


\[ \text{Mass of elementary portion} = \rho h(r \delta r \delta \theta) \quad h \text{ is the uniform thickness of sheet.} \]

Suppose \( \delta u \) is the temperature rise in the elementary portion during \( \delta t \).

\[ \therefore \text{Rate of increase of heat content in the elementary portion} \]

\[ \frac{\partial u}{\partial t} = \frac{L}{\delta t} \rho h r \delta r \delta \theta s \frac{\delta u}{\delta t} \quad s \text{ is the specific heat} \]

\[ = \rho h s r \delta r \delta \theta \frac{\partial u}{\partial t} \quad \ldots (i) \]

If \( Q_1, Q_2, Q_3 \) and \( Q_4 \) are the heat flow across the sides of the elementary portion through the edge \( PP', QQ, PQ \) and \( P'Q' \) respectively then

\[ Q_1 = -k \left( \frac{1}{r} \frac{\partial u}{\partial \theta} \right) h \delta r \quad k \text{ is thermal conductivity} \]

\[ Q_2 = -k \left( \frac{1}{r} \frac{\partial u}{\partial \theta} \right)_{\theta+\delta \theta} h \delta r \]

\[ Q_3 = -k \left( \frac{\partial u}{\partial r} \right)_{r} h r \delta \theta \]

\[ Q_4 = -k \left( \frac{\partial u}{\partial r} \right)_{r+\delta r} h (r + \delta r) \delta \theta \]

The rate of increase of heat \( (Q_1 - Q_2 + Q_3 - Q_4) \) in the elementary portion is equal to the equation \((i)\).

\[ \therefore \quad \rho h s r \delta r \delta \theta \frac{\partial u}{\partial t} = k h \left[ \left( \frac{1}{r} \frac{\partial u}{\partial \theta} \right)_{\theta+\delta \theta} - \left( \frac{1}{r} \frac{\partial u}{\partial \theta} \right)_{\theta} \right] \delta r \]

\[ + \left[ \left( \frac{\partial u}{\partial r} \right)_{r+\delta r} - \left( \frac{\partial u}{\partial r} \right)_{r} \right] r \delta \theta + \delta r \delta \theta \left( \frac{\partial u}{\partial r} \right)_{r+\delta r} \]

\[ \therefore \quad \frac{\partial u}{\partial t} = k \rho s \left[ \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right]_{\theta+\delta \theta} + \left( \frac{\partial u}{\partial r} \right)_{r+\delta r} - \left( \frac{\partial u}{\partial r} \right)_{r} + \frac{1}{r} \frac{\partial u}{\partial r} \right]_{r+\delta r} \]

Now taking limit \( \delta r \to 0 \) and \( \delta \theta \to 0 \), we get

\[ \frac{\partial u}{\partial t} = k \rho s \left[ \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial}{\partial r} \left( \frac{\partial u}{\partial r} \right) + \frac{1}{r} \frac{\partial u}{\partial r} \right] \]
\[ u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} + \frac{1}{r} \frac{\partial u}{\partial r} = 0 \]

This is the heat equation in polar coordinates.

In steady state \( \frac{\partial u}{\partial t} = 0 \)

\[ \Rightarrow \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \]

This is called Laplace equation in polar coordinates.

### 5.6.3 Solution of Two Dimensional Heat Equation (Laplace Equation) in Polar Coordinates (Steady State)

We have

\[ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \]

\( \Rightarrow \frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \) \hspace{1cm} ... (i)

Let \( u(r, \theta) = R(r) \cdot \phi(\theta) \) \hspace{1cm} ... (ii)

Putting the value of \( u \) in equation (i), we get

\[ r^2 \phi \frac{d^2 R}{dr^2} + r \phi \frac{dR}{dr} + R \frac{d^2 \phi}{d\theta^2} = 0 \]

\[ \Rightarrow \left( r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} \right) \phi + R \frac{d^2 \phi}{d\theta^2} = 0 \]

\[ \Rightarrow \frac{r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr}}{R} = -\frac{1}{\phi} \frac{d^2 \phi}{d\theta^2} = k, \text{ } k \text{ is a constant.} \]

Taking first and last member, we get

\[ r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} - Rk = 0. \]

which is linear homogeneous differential equation of second order so put \( r = e^z \), we get the solution of above equation as

\[ R = c_1 r^{\sqrt{k}} + c_2 r^{-\sqrt{k}}. \]
Next, taking second and last member, we get
\[ \frac{d^2 \phi}{d \theta^2} + k \phi = 0 \quad \Rightarrow \quad \phi = c_3 \cos \sqrt{k} \theta + c_4 \sin \sqrt{k} \theta \]

Hence from equation (ii), we get
\[ u(r, \theta) = (c_1 r^\sqrt{k} + c_2 e^{-\sqrt{k}r})(c_3 \cos \sqrt{k} \theta + c_4 \sin \sqrt{k} \theta) \quad \ldots (iii) \]

**Case I:** If \( k = p^2 \), then from (iii), we have
\[ u(r, \theta) = (c_1 r^p + c_2 r^{-p})(c_3 \cos p\theta + c_4 \sin p\theta) \]

**Case II:** If \( k = -p^2 \), then
\[ u(r, \theta) = \{c_1 \cos(p \log r) + c_2 \sin(p \log r)\} \{c_3 e^{p\theta} + c_4 e^{-p\theta}\} \]

**Case III:** If \( k = 0 \), then
\[ u(r, \theta) = \{c_1 \log r + c_2\} \{c_3 \theta + c_4\} \]

The choice of most suitable solution among these solutions depends upon the physical nature of the problem and the boundary conditions.

**Example 31:** A thin semi circular plate of radius \( a \) has its boundary diameter kept at 0°C and its circumference at 100°C. If \( u(r, \theta) \) is the steady state temperature distribution, find \( u\left(\frac{a}{2}, \frac{\pi}{2}\right) \)

**Solution:** The steady state temperature distribution \( u(r, \theta) \) is a solution of
\[ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \quad \ldots (1) \]

The boundary conditions are
(i) \( u(r, 0) = u(r, \pi) = 0, \quad 0 < r < a \)
and (ii) \( u(a, \theta) = f(\theta) = 100, \quad 0 < \theta < 2\pi \).

The solution of equation (1) which suits above boundary conditions is (in Case I)
so
\[ u(r, \theta) = (c_1 r^p + c_2 r^{-p})(c_3 \cos p\theta + c_4 \sin p\theta) \quad \ldots (2) \]

At \( \theta = 0, \ u = 0 \), from (2), we get
\[ 0 = (c_1 r^p + c_2 r^{-p})c_3 \quad \Rightarrow \ c_3 = 0 \quad (c_1 r^p + c_2 r^{-p} \neq 0) \]

\[ \therefore \text{From equation (2), we get} \]
\[ u(r, \theta) = (c_1 r^p + c_2 r^{-p})c_4 \sin p\theta \quad \ldots (3) \]

At \( \theta = \pi, \ u = 0 \)
\[ 0 = (c_1 r^p + c_2 r^{-p})c_4 \sin p\pi \quad \Rightarrow \ c_4 \sin p\pi = 0 = \sin n\pi \quad \Rightarrow \ (p = n) \]

\[ \therefore \quad (3) \text{ becomes} \]
\[ u(r, \theta) = (c_1 r^n + c_2 r^{-n})c_4 \sin n\theta \quad \ldots (4) \]
APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

Now as $r \to 0$, $u(r, \theta)$ must be finite and hence $c_2 = 0$

\[
\therefore \quad u(r, \theta) = c_1 c_2 r^n \sin n\theta
\]

or

\[
\therefore \quad u(r, \theta) = \sum_{n=1}^{\infty} b_n r^n \sin n\theta \quad (c_1 c_2 = b_n) \quad \ldots (5)
\]

Putting $r = a$ and $u = 100$

\[
100 = \sum_{n=1}^{\infty} b_n a^n \sin n\theta
\]

\[
\therefore \quad b_n a^n = \frac{2}{\pi} \int_{0}^{\pi} 100 \sin n\theta d\theta = \frac{200}{n\pi} \left[ -\cos n\theta \right]_{0}^{\pi} = \frac{200}{n\pi} [1 - \cos n\pi]
\]

\[
= \frac{200}{n\pi} [1 - (-1)^n]
\]

\[
\Rightarrow \quad b_n = \frac{200}{n\pi a^n} \left( 1 - (-1)^n \right) = \begin{cases} 
\frac{400}{n\pi a^n}, & \text{if } n \text{ is odd} \\
0, & \text{if } n \text{ is even}
\end{cases}
\]

From (5),

\[
u(r, \theta) = \frac{400}{\pi} \sum_{n=0}^{\infty} \frac{1}{n} \left( \frac{r}{a} \right)^n \sin n\theta = \frac{400}{\pi} \sum_{m=1}^{\infty} \frac{1}{(2m-1)} \left( \frac{r}{a} \right)^{2m-1} \sin(2m-1)\theta
\]

(putting, $n = 2m-1$)

\[
\therefore \quad u\left( a, \frac{\pi}{4}, \frac{\pi}{2} \right) = \frac{400}{\pi} \sum_{m=1}^{\infty} \frac{1}{(2m-1)} \left( \frac{1}{4} \right)^{2m-1} \sin \left( \frac{(2m-1)\pi}{2} \right). \quad \text{Ans.}
\]

**EXERCISE 5.4**

1. The bounding diameter of a semi circular plate of radius 10 cm is kept 0°C and the temperature along the semi circular boundary given by

\[
u(10, \theta) = \begin{cases} 
50\theta, & \text{when } 0 \leq \theta \leq \frac{\pi}{2} \\
50(\pi - \theta), & \text{when } \frac{\pi}{2} \leq \theta < \pi.
\end{cases}
\]

Find the steady state temperature $u(r, \theta)$ at any point.

\[
\text{Ans. } u(r, \theta) = \frac{200}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{(2m-1)^2} \left( \frac{r}{10} \right)^{2m-1} \sin(2m-1)\theta
\]
2. Determine the steady state temperature at the points in the sector given by $0 \leq \theta \leq \frac{\pi}{4}, 0 \leq r \leq a$ of a circular plate if the temperature is maintained at 0°C along the side edges and at a constant temperature $k$°C along the curved edge.

\[
\text{Ans. } u(r, \theta) = \frac{4k}{\pi} \sum_{m=1}^{\infty} \frac{1}{2m-1} \left(\frac{r}{a}\right)^{2m-1} \sin[4(2m-1)\theta]
\]

3. A thin semi circular plate of radius $a$ has its boundary diameter kept at 0°C and its circumference at $k$°C where $k$ is a constant. Find the temperature distribution in the steady state.

\[
\text{Ans. } u(r, \theta) = \frac{4k}{\pi} \sum_{m=1}^{\infty} \frac{1}{2m-1} \left(\frac{r}{a}\right)^{2m-1} \sin(2m-1)\theta
\]

4. A plate with insulated surfaces has the shape of quadrant of a circle of radius 10 cm. The boundary radii $\theta = 0$ and $\theta = \frac{\pi}{2}$ are kept 0°C and the temperature along the circular quadrant is kept at 100 ($\pi \theta - 2\theta^2$). Find the steady state temperature distribution.

\[
\text{Ans. } u(r, \theta) = \frac{400}{\pi} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^3} \left(\frac{r}{10}\right)^{2(2m-1)} \sin[2(2m-1)\theta]
\]

5. Find the steady state temperature in a circular plate of radius $a$ which has one half of its circumference at 0°C and the other half at 100°C.

\[
\text{Ans. } u(r, \theta) = 50 + \frac{200}{\pi} \sum_{m=1}^{\infty} \frac{1}{(2m-1)} \left(\frac{r}{a}\right)^{2m-1} \sin(2m-1)\theta
\]

### 5.7 WAVE EQUATION IN TWO DIMENSION

Consider a tightly stretched uniform membrane in $xy$-plane having the same tension $T$ at every point. Since the deflection $u(x, y, t)$ is in the perpendicular direction to $xy$-plane.

The vertical component of $T \delta y = T \delta y \sin \beta - T \delta y \sin \alpha$

\[= T \delta y (\tan \beta - \tan \alpha) \quad \text{approximately, since} \alpha \text{and} \beta \text{are very small}
\]

\[= T \delta y \left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_{x} = T \delta y \frac{\partial^2 u}{\partial x^2}\delta x
\]

Similarly, the vertical component due to the force $T \delta x$

\[= T \delta x \frac{\partial^2 u}{\partial y^2}\delta y
\]
Hence the equation of motion of the element \(ABCD\) is

\[
m \delta x \delta y \frac{\partial^2 u}{\partial t^2} = T \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \delta x \delta y
\]

or

\[
\frac{\partial^2 u}{\partial t^2} = \frac{T}{m} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)
\]

\[
\Rightarrow \frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right); \quad c^2 = \frac{T}{m}.
\]

This is two dimensional wave equation.

### 5.7.1 Solution of Two Dimensional Wave Equation

We have

\[
\frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad \text{...(i)}
\]

Let

\[
u(x, y, t) = X(t) Y(y) T(t) \quad \text{...(ii)}
\]

\[
\Rightarrow \frac{\partial^2 u}{\partial t^2} = XY \frac{d^2 T}{dt^2}, \quad \frac{\partial^2 u}{\partial x^2} = Y T \frac{d^2 X}{dx^2} \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = X T \frac{d^2 Y}{dy^2}
\]

Putting these values in equation (i), we get

\[
XY \frac{d^2 T}{dt^2} = c^2 \left( YT \frac{d^2 X}{dx^2} + XT \frac{d^2 Y}{dy^2} \right)
\]
\[
\frac{1}{c^2T} \frac{d^2T}{dt^2} = \frac{1}{X} \frac{d^2X}{dx^2} + \frac{1}{Y} \frac{d^2Y}{dy^2} \quad \text{(dividing by } XYT) \]

\[
\frac{1}{c^2T} \frac{d^2T}{dt^2} = \frac{1}{X} \frac{d^2X}{dx^2} + \frac{1}{Y} \frac{d^2Y}{dy^2} = k \quad \text{(iii)}
\]

Let
\[
\frac{1}{X} \frac{d^2X}{dx^2} = k_1 \Rightarrow \frac{d^2X}{dx^2} - k_1 X = 0
\]

\[
X = c_1 e^{\sqrt{k_1} x} + c_2 e^{-\sqrt{k_1} x}
\]

and again let
\[
\frac{1}{Y} \frac{d^2Y}{dy^2} = k_2 \Rightarrow \frac{d^2Y}{dy^2} - k_2 Y = 0
\]

\[
Y = c_3 e^{\sqrt{k_2} y} + c_4 e^{-\sqrt{k_2} y}
\]

Finally
\[
\frac{1}{c^2T} \frac{d^2T}{dt^2} = k \Rightarrow \frac{d^2T}{dt^2} = kc^2 T \Rightarrow \frac{d^2T}{dt^2} - kc^2 T = 0
\]

\[
T = c_5 e^{\sqrt{k_3} t} + c_6 e^{-\sqrt{k_3} t}
\]

From equation (iii), we get
\[
\frac{d^2}{dt^2} T = kc^2 T
\]

where \( k = k_1 + k_2 \).

There arise the following cases:

**Case I:** If \( k = 0 \), then
\[
\frac{d^2}{dt^2} T = 0
\]

\[
u(x, y, t) = (c_1 x + c_2)(c_3 + c_4)(c_5 t + c_6)
\]

**Case II:** If \( k = p^2 \), then
\[
\frac{d^2}{dt^2} T = p^2 c^2 T
\]

\[
u(x, y, t) = (c_1 e^{p_1 x} + c_2 e^{-p_1 x})(c_3 e^{p_2 y} + c_4 e^{-p_2 y})(c_5 e^{p_3 t} + c_6 e^{-p_3 t})
\]

where \( p^2 = p_1^2 + p_2^2 \).

**Case III:** If \( k = -p^2 \), then
\[
\frac{d^2}{dt^2} T = -p^2 c^2 T
\]

\[
u(x, y, t) = (c_1 \cos p_1 x + c_2 \sin p_1 x)(c_3 \cos p_2 y + c_4 \sin p_2 y)(c_5 \cos p_3 t + c_6 \sin p_3 t)
\]

where \( p^2 = p_1^2 + p_2^2 \).

Since the physical nature of the problem for a rectangular membranes \((0, 0), (a, 0), (a, b), (0, b)\) is periodic so the solution in case III, will be consistent with the physical nature of the problem. Here we will consider case III solution.

**Example 32:** Find the deflection \( u(x, y, t) \) of a rectangular membranes \((0 \leq x \leq a, 0 \leq y \leq b)\) whose boundary is fixed given that it starts from rest and \( u(x, y, 0) = xy (a - x) (b - y) \).
**Solution:** The vibrations of the membrane are governed by the wave equation

\[
\frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)
\]  

...(1)

The physical nature of the problem is periodic so, we consider solution of equation (1) in case III.

\[
 u(x, y, t) = \sin \left( \frac{m\pi}{a} \right) \sin \left( \frac{n\pi}{b} \right) \]  

where \( p^2 = p_1^2 + p_2^2 \).

The boundary conditions are

(i) \( u(0, y, t) = u(a, y, t) = 0 \)

(ii) \( u(x, 0, t) = u(x, b, t) = 0 \)

and the initial condition are

(iii) \( u(x, y, 0) = xy (a - x) (b - y) \)  

(iv) \( \frac{\partial u}{\partial t} = 0, \) at \( t = 0 \)

Using boundary conditions in (2), we get

At \( x = 0, \) the temperature the defelection \( u = 0 \)

\( \Rightarrow \quad 0 = c_1(c_3 \cos p_2 y + c_4 \sin p_2 y)(c_5 \cos c\pi t + c_6 \sin c\pi t) \)

\( \Rightarrow \quad c_1 = 0 \) [(otherwise \( u(x, y, t) = 0 \)) which is impossible]

From (2), we obtain

\[
 u(x, y, t) = c_2 \sin p_1 x(\sin c_3 x) (c_3 \cos p_2 y + c_4 \sin p_2 y)(c_5 \cos c\pi t + c_6 \sin c\pi t) \]  

...(3)

At \( x = a, u = 0 \)

\( \Rightarrow \quad 0 = c_2 \sin p_1 a(c_3 \cos p_2 y + c_4 \sin p_2 y)(c_5 \cos c\pi t + c_6 \sin c\pi t) \)

\( \Rightarrow \quad \sin p_1 a = 0 \quad \Rightarrow \quad \sin p_1 a = \sin m\pi \quad \Rightarrow \quad p_1 = \frac{m\pi}{a} \).

At \( y = 0, u = 0 \)

(3) gives

\( 0 = c_2 \sin p_1 x(c_3 + 0)(c_5 \cos c\pi t + c_6 \sin c\pi t) \)

\( \Rightarrow \quad c_3 = 0 \) [(otherwise, \( u = 0 \)) which is impossible]

Putting \( c_3 = 0 \) in equation (3), we get

\[
 u(x, y, t) = c_2 c_4 \sin p_1 x \sin p_2 y(c_5 \cos c\pi t + c_6 \sin c\pi t) \]  

...(4)

At \( y = b, u = 0 \)
Equation (4) gives
\[ 0 = c_2c_4 \sin p_1 x \sin p_2 b (c_5 \cos cp t + c_6 \sin cp t) \]
\[ \Rightarrow \quad \sin p_2 b = 0 \quad [\text{otherwise } u(x, y, t) = 0, \text{ which is impossible}] \]
\[ \Rightarrow \quad \sin p_2 b = \sin n\pi \quad \Rightarrow \quad p_2 = \frac{n\pi}{b}. \]
Putting the values of \( p_1 \) and \( p_2 \) in equation (4), we get
\[ u(x, y, t) = c_2 c_4 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} (c_5 \cos cp t + c_6 \sin cp t) \quad \ldots(5) \]
Putting \( \frac{\partial u}{\partial t} = 0 \) and \( u = 0 \), we get
\[ \Rightarrow \quad \left( \frac{\partial u}{\partial t} \right)_{t=0} = 0 = c_2 c_4 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} (-c_5 c p \sin cp (0) + c_6 p \cos cp (0)) \]
\[ \Rightarrow \quad 0 = c_2 c_4 c_6 c p \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \Rightarrow \quad c_6 = 0 \quad [\text{otherwise } u = 0] \]
\[ \therefore \quad \text{Equation (5), gives} \]
\[ u(x, y, t) = c_2 c_4 c_5 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos cp t \]
The general form of solution is
\[ u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos cp t \quad \ldots(6) \]
where \( A_{mn} = c_2 c_4 c_5 \) and \( p_{mn} = \pi \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \).

At \( t = 0 \), from (6), we get
\[ xy(a - x)(b - y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \cdot \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \]
which is double half range Fourier sine series.
\[ \therefore \quad A_{mn} = \frac{2}{a} \cdot \frac{2}{b} \int_{x=0}^{a} \int_{y=0}^{b} xy(a - x)(b - y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dxdy \]
\[ \quad = \frac{4}{ab} \int_{x=0}^{a} x(a - x) \sin \frac{m\pi x}{a} \, dx \int_{y=0}^{b} y(b - y) \sin \frac{n\pi y}{b} \, dy \]
\[
\begin{align*}
\frac{4}{ab} & \left[ -x(a-x) \cos \frac{m\pi x}{a} + (a-2x) \frac{a^2}{m^2 \pi^2} \sin \frac{m\pi x}{a} - \frac{2a^3}{m^3 \pi^3} \cos \frac{m\pi x}{a} \right]_0^a \\
\times & \left[ -y(b-y) \frac{b}{n\pi} \cos \frac{n\pi y}{b} + (b-2y) \frac{b^2}{n^2 \pi^2} \sin \frac{n\pi y}{b} - \frac{2b^3}{n^3 \pi^3} \cos \frac{n\pi y}{b} \right]_0^b \\
= & \frac{4}{ab} \left( \frac{-2a^3}{m^3 \pi} \cos m\pi + \frac{2a^3}{m^3 \pi} \right) \left( \frac{-2b^3}{n^3 \pi} \cos n\pi + \frac{2b^3}{n^3 \pi} \right) \\
= & \frac{4}{ab} \frac{2a^3}{m^3 \pi} \frac{2b^3}{n^3 \pi} \left\{ (-1)^{m+1} + 1 \right\} \left\{ (-1)^{n+1} + 1 \right\} \\
= & \frac{64a^2 b^2}{m^3 n^3 \pi^6} \text{ where } m \text{ and } n \text{ are odd.}
\end{align*}
\]

Putting the value of \( A_{mn} \) in equation (6), we get

\[
u(x, y, t) = \sum_{m=1,3,5,\ldots}^{\infty} \sum_{n=1,3,5,\ldots}^{\infty} \frac{64a^2 b^2}{m^3 n^3 \pi^6} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos c m n t
\]

\[
= \frac{64a^2 b^2}{\pi^6} \sum_{m=1,3,5,\ldots}^{\infty} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{m^3 n^3} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos c m n t \text{ Ans.}
\]

where \( p_{mn} = \pi \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \).

**Exercise 5.5**

1. The initial displacement of the rectangular membrane \((0 < x < 1, 0 < y < 2)\) where boundary is fixed given that it starts from rest and \( u(x, y, 0) = xy (1-x) (2-y) \). Find \( u(x, y, t) \).

\[
\text{Ans. } u(x, y, t) = \sum \sum \frac{256}{m^3 n^3 \pi^6} \cos c m n t \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{2}
\]

where \( m \) and \( n \) both are odd and \( p_{mn} = \pi \sqrt{\frac{m^2}{4}} + \frac{n^2}{4} \).
2. The initial displacement of a rectangular membrane \((0 < x < a, \ 0 < y < b)\) whose boundary is fixed is given by \(f(x, y) = \kappa (ax - x^2) (by - y^2)\) show that

\[
u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}
\]

where \(A_{mn} = \frac{16\kappa a^2 b^2 (1 - \cos m\pi)(1 - \cos n\pi)}{m^2 n^2 \pi^6} \)

and \(p_{mn}^2 = \pi^2 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \).

3. Find the deflection \(u(x, y, t)\) of the square membrane with \(a = b = c = 1\), if the initial velocity is zero and the initial deflection \(f(x, y) = A \sin \pi x \sin 2\pi y\).

\[
[u(x, y, t)] = \frac{\partial u}{\partial t}
\]

\(\text{Ans.} \ u(x, y, t) = A \cos \sqrt{5}\pi x \sin \pi x \sin 2\pi y\]

4. Find the deflection \(u(x, y, t)\) of a rectangular membrane \((0 \leq x \leq a, \ 0 \leq y \leq b)\) and

\[
\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0. \ \text{The deflection at} \ t = 0 \ \text{is} \ xy (a^2 - x^2) (b^2 - y^2).
\]

\[
\text{Ans.} \ u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{144\kappa a^2 b^2}{m^2 n^2 \pi^6} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}
\]

where \(p_{mn}^2 = \pi^2 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \).

5. Find the deflection \(u(x, y, t)\) of a membrane \((0 \leq x \leq a, \ 0 \leq y \leq b)\) when the initial velocity is zero and the initial deflection is given by \(\frac{2\pi x}{a} \frac{\pi y}{b} \).

\[
\text{Ans.} \ u(x, y, t) = \sin \frac{2\pi x}{a} \frac{\pi y}{b} \cdot \cos \pi ct \sqrt{\frac{4}{a^2} + \frac{1}{b^2}}
\]

5.8 EQUATIONS OF TRANSMISSION LINES

Suppose the resistance, inductance and capacitance vary linearly with \(x\) in transmission line, where \(x\) is the distance from the source of electricity, measured along the line. If \(V(x, t)\) denote the potential at a point on the cable, \(i(x, t)\) is current there \(R\) is the resistance of the cable per unit length and \(L\) is inductance per unit length.

Then the equations

\[
\frac{\partial V}{\partial x} = -L \frac{\partial i}{\partial t} \frac{\partial i}{\partial x} = -C \frac{\partial V}{\partial t}
\]
APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

\[ \frac{\partial^2 V}{\partial x^2} = RC \frac{\partial V}{\partial t} \]
\[ \frac{\partial^2 i}{\partial x^2} = RC \frac{\partial i}{\partial t} \]  
...(A)

and

\[ \frac{\partial^2 V}{\partial x^2} = LC \frac{\partial^2 V}{\partial t^2} \]
\[ \frac{\partial^2 i}{\partial x^2} = LC \frac{\partial^2 i}{\partial t^2} \]  
...(B)

The set (A) is known as ‘Telegraph equations’, and (B) is known as ‘radio equations’.

**Example 33**: Neglecting \( R \) and \( G \) obtain potential \( V(x, t) \) in a line \( l \) km. long, after \( t \) seconds the ends are suddenly grounded, if initially \( i(x, 0) = I_0 \) and \( V(x, 0) = V_0 \sin \frac{\pi x}{l} \). Also current,

\( \text{(U.P.T.U. 2008)} \)

**Solution**: We have

\[ \frac{\partial^2 V}{\partial x^2} = LC \frac{\partial^2 V}{\partial t^2} \]  
...(1)

Let

\[ V(x, t) = X(x)T(t) \]

\[ \therefore \]

\[ \frac{\partial^2 V}{\partial x^2} = T \frac{d^2 X}{dx^2} \quad \text{and} \quad \frac{\partial^2 V}{\partial t^2} = X \frac{d^2 T}{dt^2} \]

Putting these values in equation (1), we obtain

\[ T \frac{d^2 X}{dx^2} = LCX \frac{d^2 T}{dt^2} \]

\[ \Rightarrow \frac{1}{X} \frac{d^2 X}{dx^2} = \frac{LC}{T} \frac{d^2 T}{dt^2} = -p^2 \]

\[ \therefore \]

\[ \frac{1}{X} \frac{d^2 X}{dx^2} = -p^2 \]

\[ \Rightarrow \frac{d^2 T}{dt^2} + p^2 T = 0 \]

\[ \Rightarrow \]

\[ T = c_3 \cos \frac{pt}{\sqrt{LC}} + c_4 \sin \frac{pt}{\sqrt{LC}} \]

From equation (2), we get

\[ V(x, t) = (c_1 \cos px + c_2 \sin px) \left( c_3 \cos \frac{pt}{\sqrt{LC}} + c_4 \sin \frac{pt}{\sqrt{LC}} \right) \]  
...(3)
The boundary conditions are

(i) \( V(0, t) = 0 \)  
(ii) \( V(l, t) = 0 \)  

As the ends are suddenly grounded.

The initial conditions are

(iii) \( V(x, 0) = V_0 \sin \frac{\pi x}{l} \Rightarrow \frac{\partial V(x, 0)}{\partial t} = 0 \)

(iv) \( i(x, 0) = I_0 \Rightarrow \frac{\partial i(x, 0)}{\partial t} = 0 \)

At \( x = 0, V = 0 \), equation (3) gives

\[
0 = c_1 \left( c_3 \cos \frac{pt}{\sqrt{LC}} + c_4 \sin \frac{pt}{\sqrt{LC}} \right)
\]

\[
\Rightarrow c_1 = 0 \text{ [otherwise } V(x, t) = 0, \text{ which is impossible]} \]

\[
\therefore \text{ From (3), we have }
\]

\[
V(x, t) = c_2 \sin p x \left( c_3 \cos \frac{pt}{\sqrt{LC}} + c_4 \sin \frac{pt}{\sqrt{LC}} \right) \quad \text{(4)}
\]

At \( x = l, V = 0 \), equation (4) gives

\[
p = \frac{n\pi}{l}
\]

\[
\therefore \text{ From equation (4), we have }
\]

\[
V(x, t) = \sin \frac{n\pi x}{l} \left( c_2 c_3 \cos \frac{n\pi t}{l\sqrt{LC}} + c_2 c_4 \sin \frac{n\pi t}{l\sqrt{LC}} \right)
\]

The general form of above solution is

\[
V(x, t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} \left( a_n \cos \frac{n\pi t}{l\sqrt{LC}} + b_n \sin \frac{n\pi t}{l\sqrt{LC}} \right) \quad \text{(5)}
\]

\[
\frac{\partial V}{\partial t} = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} \left( -a_n \frac{l\sqrt{LC}}{n\pi} \cos \frac{n\pi t}{l\sqrt{LC}} + b_n \frac{l\sqrt{LC}}{n\pi} \sin \frac{n\pi t}{l\sqrt{LC}} \right)
\]

At \( t = 0, \frac{\partial V}{\partial t} = 0 \)

\[
\therefore \text{ Above equation gives }
\]

\[
0 = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} \cdot b_n \frac{l\sqrt{LC}}{n\pi} \Rightarrow b_n = 0.
\]

From (5), we obtain

\[
V(x, t) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \cos \frac{n\pi t}{l\sqrt{LC}} \quad \text{(6)}
\]

and at \( t = 0, V = V_0 \sin \frac{\pi x}{l} \)
APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

\[ V_0 \sin \frac{\pi x}{l} = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \]

or

\[ V_0 \sin \frac{\pi x}{l} = a_1 \sin \frac{\pi x}{l} + a_2 \sin \frac{2\pi x}{l} + \ldots \]

\[ \Rightarrow \quad a_1 = V_0 \]

and \( a_2 = a_3 = a_4 = \ldots = a_n = 0 \).

Putting the value of \( a_1 \) in equation (6), we get

\[ V(x, t) = V_0 \sin \frac{\pi x}{l} \cos \frac{\pi t}{l\sqrt{LC}}. \quad \text{Ans.} \]

Second part: We know that

\[ \frac{\partial V}{\partial x} = -L \frac{\partial i}{\partial t} \]

So

\[ \frac{\partial}{\partial x} \left( V_0 \sin \frac{\pi x}{l} \cos \frac{\pi t}{l\sqrt{LC}} \right) = -L \frac{\partial i}{\partial t} \]

\[ \Rightarrow \quad \frac{\pi}{l} V_0 \cos \frac{\pi x}{l} \cos \frac{\pi t}{l\sqrt{LC}} = -L \frac{\partial i}{\partial t} \quad \ldots(7) \]

Integrating (7), we obtain

\[ i = -V_0 \sqrt{\frac{C}{L}} \cos \frac{\pi x}{l} \sin \frac{\pi t}{l\sqrt{LC}} + \phi(x) \quad \ldots(8) \]

But at \( t = 0, i = I_0 \), equation (8), gives

\[ I_0 = \phi(x) \]

Putting the value of \( \phi(x) \) in equation (8), we get

\[ i = I_0 - V_0 \sqrt{\frac{C}{L}} \cos \frac{\pi x}{l} \sin \frac{\pi t}{l\sqrt{LC}}. \quad \text{Ans.} \]

**EXERCISE 5.6**

1. Find the potential \( V(x, t) \) in a line of length \( l \), after \( t \) seconds the ends were suddenly grounded, given that \( i(x, 0) = i_0 \) and \( V(x, 0) = V_1 \sin \frac{\pi x}{l} + V_5 \sin \frac{5\pi x}{l} \).

\[
\begin{bmatrix}
\text{Ans.} \quad V(x, t) = V_1 \sin \frac{\pi x}{l} \cos \frac{\pi t}{l\sqrt{LC}} + V_5 \sin \frac{5\pi x}{l} \cos \frac{5\pi t}{l\sqrt{LC}}
\end{bmatrix}
\]
2. Neglecting $R$ and $G$ find the e.m.f. $e(x, t)$ in a line one km long, $t$ seconds after the ends were suddenly grounded, if initially $i(x, 0) = I$ so that \( \frac{de}{dt} = 0 \) at $t = 0$ and $e(x, 0)$

\[
E_1 \sin \frac{\pi x}{a} + E_2 \sin \frac{2\pi x}{a}
\]

\[\text{Ans. } e(x, t) = E_1 \sin \frac{\pi x}{a} \cos \frac{\pi t}{\alpha \sqrt{LC}} + E_2 \sin \frac{2\pi x}{a} \cos \frac{2\pi t}{\alpha \sqrt{LC}}\]

3. Obtain the solution of the radio equation \( \frac{\partial^2 V}{\partial x^2} = LC \frac{\partial^2 V}{\partial t^2} \) when a periodic e.m.f. $V_0 \cos pt$ is applied at the end $x = 0$ of the line.

\[\text{Ans. } V(x, t) = V_0 \cos \left( pt - px\sqrt{LC} \right)\]

4. A transmission line 1000 km. long is initially under steady state conditions with potential 1300 volts at the sending end ($x = 0$) and 1200 volts at the receiving end ($x = 1000$). The terminal end of the line is suddenly grounded, but the potential at the source is kept at 1300 volts. Assuming the inductance and leakance to be negligible find the potential $V(x, t)$.

\[\text{Hint: Apply } \frac{\partial^2 V}{\partial x^2} = R \frac{\partial E}{\partial t}; \text{ in steady state } \frac{\partial E}{\partial t} = 0\]

\[\text{Ans. } V(x, t) = 1300 - 1 \cdot 3x + \frac{2400}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l} e^{-\frac{n^2 \pi^2 t}{l^2 RC}}\]

5. A steady voltage distribution of 20 volts at the sending end and 12 volts at the receiving end is maintained in a telephone wire of length $l$. At time $t = 0$, the receiving end is grounded. Find the voltage and current $t$ second later. Neglect leakage and inductance.

\[\text{Ans. } V(x, t) = \frac{20(l - x)}{l} + \frac{24}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l} e^{-\frac{n^2 \pi^2 t}{l^2 RC}}\]

\[i(x, t) = \frac{20}{lR} + \frac{24}{lR} \sum_{n=1}^{\infty} (-1)^n \cos \frac{n\pi x}{l} e^{-\frac{n^2 \pi^2 t}{l^2 RC}}\]