CHAPTER 1

Design of Curved Beams

NOTATIONS AND SYMBOLS USED

\( \sigma \) = Stress, MPa.
\( \sigma_d \) = Direct stress, tensile or compressive, MPa.
\( \sigma_i \) = Stress at inner fibre, MPa.
\( \sigma_o \) = Stress at outer fibre, MPa.
\( \sigma_{bi} \) = Normal stress due to bending at inner fibre, MPa.
\( \sigma_{bo} \) = Normal stress due to bending at outer fibre, MPa.
\( M_b \) = Bending moment for critical section, N-mm.
\( c_i \) = Distance of neutral axis from inner fibre, mm.
\( c_o \) = Distance of neutral axis from outer fibre, mm.
\( e \) = Eccentricity, mm.
\( r_i \) = Distance of inner fibre from centre of curvature, mm.
\( r_o \) = Distance of outer fibre from centre of curvature, mm.
\( r_c \) = Distance of centroidal axis from centre of curvature, mm.
\( r_n \) = Distance of neutral axis from centre of curvature, mm.
\( d \) = Diameter of circular rod used in curved beam, mm.
\( h \) = Depth of curved beam [square, rectangular, trapezoidal or I-section], mm.
\( \tau_{max} \) = Maximum shear stress, MPa.
\( A \) = Area of cross-section of member, (curved beam), mm\(^2\).
\( P \) = Load on member, N.

INTRODUCTION

Machine frames having curved portions are frequently subjected to bending or axial loads or to a combination of bending and axial loads. With the reduction in the radius of curved portion, the stress due to curvature become greater and the results of the equations of straight beams when used becomes less satisfactory. For relatively small radii of curvature, the actual stresses
may be several times greater than the value obtained for straight beams. It has been found from
the results of Photoelastic experiments that in case of curved beams, the neutral surface does
not coincide with centroidal axis but instead shifted towards the centre of curvature. It has also
been found that the stresses in the fibres of a curved beam are not proportional to the distances
of the fibres from the neutral surfaces, as is assumed for a straight beam.

**Differences between Straight and Curved Beams**

<table>
<thead>
<tr>
<th>Straight beam</th>
<th>Curved beam</th>
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<tbody>
<tr>
<td>1. The neutral axis of beam coincides with centroidal axis.</td>
<td>The neutral axis is shifted towards the centre of curvature by a distance called eccentricity i.e. the neutral axis lies between centroidal axis and centre of curvature.</td>
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<td>2. The variation of normal stress due to bending is linear, tensile at the inner fibre and compressive at the outer fibre with zero value at the centroidal axis.</td>
<td>The variation of normal stress due to bending across section is non-linear and is hyperbolic.</td>
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**Derivation of Expression to Determine Stress at any Point on the Fibres of a Curved Beam**

Consider a curved beam with \( r_c \), as the radius of centroidal axis, \( r_n \), the radius of neutral surface, \( r_i \), the radius of inner fibre, \( r_o \), the radius of outer fibre having thickness ‘h’ subjected to bending moment \( M_b \).

Let \( AB \) and \( CD \) be the two adjacent cross-sections separated from each other by a small angle \( d\phi \).

Because of \( M_b \) the section \( CD \) rotates through a small angle \( d\alpha \). The unit deformation of any fibre at a distance \( y \) from neutral surface is

\[
\text{Deformation} \quad \epsilon = \frac{\delta}{l} = \frac{y d\alpha}{(r_n - y)d\phi} \quad \ldots \ (1.1)
\]

The unit stress on this fibre is,

\[
\text{Stress} = \text{Strain} \times \text{Young’s modulus of material of beam} \quad \sigma = \epsilon E = \frac{Ey d\alpha}{(r_n - y)d\phi} \quad \ldots \ (1.2)
\]

For equilibrium, the summation of the forces acting on the cross-sectional area must be zero.

\[
i.e., \quad \int \sigma dA = 0
\]

or

\[
\int \frac{Ey d\alpha dA}{(r_n - y)d\phi} = 0
\]

\[
E \frac{d\alpha}{d\phi} \int \frac{y dA}{r_n - y} = 0 \quad \ldots \ (1.3)
\]
Also the external moment $M_b$ applied is resisted by internal moment. From equation 1.2 we have,

$$\int y(\sigma dA) = M$$

\[\text{i.e.,} \quad \int E \frac{y^2 dA}{(r_n - y) d\phi} = M\]

$$E \frac{d\alpha}{d\phi} \int \frac{y^2 dA}{(r_n - y)} = M \quad \ldots (1.4)$$

\[\text{i.e.,} \quad M = \frac{Ed\alpha}{d\phi} \int (-y) dA + r_n \int \frac{ydA}{(r_n - y)} \quad \ldots (1.5)\]

**Note:** In equation 1.5, the first integral is the moment of cross sectional area with respect to neutral surface and the second integral is zero from equation 1.3.

Therefore,

$$M = \left[\frac{E}{A} \int \frac{d\alpha}{d\phi} A e\right] \quad \ldots (1.6)$$

Here 'e' represents the distance between the centroidal axis and neutral axis.

\[\text{i.e.,} \quad e = r_c - r_n\]

Rearranging terms in equation 1.6, we get

$$\frac{d\alpha}{d\phi} = \frac{M}{A e E} \quad \ldots (1.7)$$

Substituting $\frac{d\alpha}{d\phi} = \frac{M}{A e E}$ in equation 1.2,

Stress

$$\sigma = \frac{Ey}{(r_n - y)} \frac{d\alpha}{d\phi} \text{ becomes}$$

![Figure: 1.1](image-url)
\[ \sigma = \frac{M y}{(r_n - y) A e E} \]

i.e.,
\[ \sigma = \frac{M y}{(r_n - y) A e} \] ...

(1.8)

From figure 1.1,
\[ r_n = v + y \]

or
\[ y = r_n - v \]

Therefore, from equation 1.3
\[ \int \frac{y dA}{(r_n - y)} = \int \frac{(r_n - v) dA}{(r_n - y)} \]

\[ = r_n \int \frac{dA}{v} - \int dA = 0 \]

\[ = r_n \int \frac{dA}{V} - A = 0 \]

or
\[ r_n = \frac{A}{\int \frac{dA}{V}} \] ...

(1.9)

Note: Since \( e = r_e - r_n \), equation 1.9 can be used to determine \( e \). Knowing the value of \( e \), equation 1.8 is used to determine the stress \( \sigma \).

**Straight and Curved Beams**

*(a) Straight beam*

Consider a straight beam having moment of inertia \( I \) subjected to bending moment \( M_b \) as shown in (Fig. 1.2).
From fundamental equation of bending,
\[
\frac{M_b}{I} = \frac{\sigma_b}{y} = \frac{E}{R} ; \quad \frac{M_b}{I} = \frac{\sigma_b}{y}
\]

i.e., Bending stress \( \sigma_b = \frac{M_b}{I} y \)

For a given beam, \( M_b \) and \( T \) are constant and hence \( \sigma_b \propto y \), i.e., the variation of bending stress is linear and is directly proportional to its distance from centre of gravity axis which coincides with neutral axis. The maximum bending (tensile) stress \( \sigma_b \) tensile at \( A \) and compressive at \( B \) are equal in magnitude at ‘A’ and ‘B’ as shown in the Fig. 1.2.

(b) Curved beam

Figure 1.3 shows a curved beam subjected to bending moment \( M_b \).

Let
- \( r_i = \) Distance of inner fibre from centre of curvature, \( C \)
- \( r_o = \) Distance of outer fibre from centre of curvature
- \( r_c = \) Distance of centroidal axis (CG axis) from centre of curvature
- \( r_n = \) Distance of neutral axis from centre of curvature

The neutral axis is shifted towards the centre of curvature by a distance called eccentricity ‘\( e \)’. The value ‘\( e \)’ should be computed very accurately since a small variation in the value of ‘\( e \)’ causes a large variation in the values of stress.

\[
i.e., \quad e = r_c - r_n
\]

\( c_i = \) Distance between neutral axis and inner fibre = \( r_n - r_i \)

\( c_o = \) Distance between outer fibre and neutral axis = \( r_o - r_n \)