UNIT I

Differential Calculus-I

1.1 INTRODUCTION

Calculus is one of the more beautiful intellectual achievements of human being. The mathematical study of change motion, growth or decay is calculus. One of the most important idea of differential calculus is derivative which measures the rate of change of given function. Concept of derivative is very useful in engineering, science, economics, medicine and computer science.

The first order derivative of $y$ denoted by $\frac{dy}{dx}$, second order derivative denoted by $\frac{d^2y}{dx^2}$, third order derivative by $\frac{d^3y}{dx^3}$ and so on. Thus by differentiating a function $y = f(x)$, $n$ times, successively, we get the $n$th order derivative of $y$ denoted by $\frac{d^n y}{dx^n}$ or $D^n (y)$ or $y_n(x)$. Thus, the process of finding the differential coefficient of a function again and again is called Successive Differentiation.

1.2 $n$th DERIVATIVES OF SOME STANDARD FUNCTIONS

Below we obtain formulas for the $n$th order derivatives of some standard functions.

(1) $n$th derivative of $e^{ax}$

Let $y = e^{ax}$. Then by differentiating $y$ successively, we obtain

\[ y_1 = \frac{dy}{dx} = ae^{ax}, \]

\[ y_2 = \frac{d^2 y}{dx^2} = a^2 e^{ax} \]

\[ y_3 = \frac{d^3 y}{dx^3} = a^3 e^{ax} \]

\[ \vdots \]

\[ y_n = \frac{d^n y}{dx^n} = a^n e^{ax} \]
Thus, we have the formula

\[ D^n(e^{ax}) = a^n e^{ax} \]  \hspace{1cm} \cdots(1)

In particular,

\[ D^n(e^x) = e^x \]  \hspace{1cm} \cdots(2)

(2) \textit{nth derivative of } \log(ax + b)

Let \( y = \log(ax + b) \). Then we find, by successive differentiation

\[
y_1 = \frac{dy}{dx} = \frac{a}{ax + b}
\]

\[
y_2 = \frac{d^2y}{dx^2} = (-1) \frac{a^2}{(ax + b)^2}
\]

\[
y_3 = \frac{d^3y}{dx^3} = (-1)(-2) \frac{a^3}{(ax + b)^3}
\]

\[
y_4 = \frac{d^4y}{dx^4} = (-1)(-2)(-3) \frac{a^4}{(ax + b)^4}
\]

\[
\vdots
\]

\[
y_n = \frac{d^ny}{dx^n} = (-1)^{n-1} \cdot \frac{(n-1)!a^n}{(ax + b)^n}
\]

Thus, we have the formula

\[ D^n[\log(ax + b)] = \frac{(-1)^{n-1} \cdot (n-1)!a^n}{(ax + b)^n} \]  \hspace{1cm} \cdots(3)

In particular,

\[ D^n[\log x] = \frac{(-1)^{n-1} \cdot (n-1)!}{x^n} \]  \hspace{1cm} \cdots(4)

(3) \textit{nth derivative of } (ax + b)^m

Let \( y = (ax + b)^m \)

Differentiating successively, we get

\[
y_1 = \frac{dy}{dx} = ma (ax + b)^{m-1}
\]

\[
y_2 = \frac{d^2y}{dx^2} = m(m - 1) \cdot a^2 (ax + b)^{m-2}
\]
\[ y_3 = \frac{d^3 y}{dx^3} = m(m-1)(m-2)ax^3(ax+b)^{m-2} \]

\[ y_n = m(m-1)(m-2)\cdots(n-1)nax^n(ax+b)^{m-n} \] ... (5)

This formula is true for all \( m \).

Following are some particular cases.

**Case (i):** Suppose \( m = n \) (a +ve integer)

In equation (5) becomes,

\[ D^n[(ax+b)^n] = n(n-1)(n-2)\cdots1 \cdot d^n(ax+b)^n = n! a^n \] ... (6)

In particular,

\[ D^n(x^n) = n! \] ... (7)

**Case (ii):** Suppose \( m \) is a positive integer and \( m > n \). Then formula (5) becomes

\[ D^n[(ax+b)^m] = \frac{m(m-1)\cdots(m-n+1)(m-n)(m-n-1)\cdots2 \cdot 1}{(m-n)(m-n-1)\cdots2 \cdot 1} \cdot d^n(ax+b)^{m-n} \]

\[ = \frac{m!}{(m-n)!} a^n(ax+b)^{m-n} \] ... (8)

In particular,

\[ D^n(x^m) = \frac{m!}{(m-n)!} x^{m-n} \] ... (9)

**Case (iii):** Suppose \( m \) is a positive integer and \( n > m \). From (6) we note that

\[ D^n[(ax+b)^m] = m! a^m \]

In differentiate further, the right-hand side gives zero.

Thus,

\[ D^n[(ax+b)^m] = 0 \quad \text{if} \quad n > m \] ... (10)

In particular,

\[ D^n(x^m) = 0 \quad \text{for} \quad n > m \] ... (11)

**Case (iv):** Suppose \( m = -1 \), in this case formula (5) becomes,

\[ D^n \left[ \frac{1}{ax+b} \right] = (-1)(-2)(-3)\cdots(-n)ax^n(ax+b)^{-1-n} \]

\[ = \frac{(-1)^n \cdot n! \cdot a^n}{(ax+b)^{n+1}} \] ... (12)
Case (p): Suppose \( m \) is a negative integer. Let us get \( m = -p \), so that \( p \) is a positive integer. Then formula (5) becomes,

\[
D^n \left[ \frac{1}{(ax + b)^p} \right] = \frac{(-1)^n \cdot p(p + 1) \cdots (p + n - 1) \cdot a^n}{(ax + b)^{n+p}}
\]

\[
= \frac{(-1)^n \cdot 1 \cdot 2 \cdots (p - 1) \cdot (p(p + 1) \cdots (p + n - 1) \cdot a^n}{(ax + b)^{n+p}}
\]

\[
= \frac{(-1)^n \cdot (p + n - 1)!}{(p - 1)!} \cdot \frac{a^n}{(ax + b)^{n+p}}
\]

\[\ldots \text{(13)}\]

In particular

\[
D^n \left[ \frac{1}{x^p} \right] = \frac{(-1)^n \cdot (p + n - 1)!}{(p - 1)!} \cdot \frac{1}{x^{p+n}}
\]

\[\ldots \text{(14)}\]

(4) \( n \)th derivative of \( \cos (ax + b) \)

Let \( y = \cos(ax + b) \).

Differentiating this successively, we get

\[
y_1 = \frac{dy}{dx} = -a \sin(ax + b) = a \cos(ax + b + \pi/2)
\]

\[
y_2 = \frac{d^2y}{dx^2} = -a^2 \sin(ax + b + \pi/2)
\]

\[
= a^2 \cos(ax + b + 2\pi/2)
\]

\[
y_3 = \frac{d^3y}{dx^3} = -a^3 \sin(ax + b + 2\pi/2)
\]

\[
= a^3 \cos(ax + b + 3\pi/2)
\]

\[
\ldots \text{......................}
\]

\[
y_n = \frac{d^ny}{dx^n} = a^n \cos(ax + b + n\pi/2)
\]

Thus, we obtain the formula

\[
D^n[\cos(ax + b)] = a^n \cos(ax + b + n\pi/2)
\]

\[\ldots \text{(15)}\]

In particular,

\[
D^n(\cos x) = \cos(x + n\pi/2)
\]

\[\ldots \text{(16)}\]

(5) \( n \)th derivative of \( \sin(ax + b) \)

Let \( y = \sin(ax + b) \)
Differentiating successively, we get
\[
\frac{dy}{dx} = y_1 = a \cos(ax + b) = a \sin(ax + b + \pi/2)
\]
\[
\frac{d^2y}{dx^2} = y_2 = a^2 \cos(ax + b + \pi/2) = a^2 \sin(ax + b + 2\pi/2)
\]
\[
\frac{d^3y}{dx^3} = y_3 = a^3 \cos(ax + b + 2\pi/2) = a^3 \sin(ax + b + 3\pi/2)
\]
\[
\vdots
\]
\[
\frac{d^n y}{dx^n} = y_n = a^n \sin(ax + b + n\pi/2)
\]

Thus, we have the formula,
\[
D^n [\sin(ax + b)] = a^n \sin(ax + b + n\pi/2)
\]  
...(17)

In particular,
\[
D^n (\sin x) = \sin(x + n\pi/2)
\]  
...(18)

(6) \(n\)th derivative of \(e^{ax} \sin (bx + c)\)

Let
\[
y = e^{ax} \sin(bx + c)
\]
\[
y_1 = \frac{dy}{dx} = ae^{ax} \sin(bx + c) + be^{ax} \cos(bx + c)
\]

For computation of higher-order derivatives it is convenient to express the constants \(a\) and \(b\) in terms of the constants \(k\) and \(\alpha\) defined by
\[
a = k \cos \alpha, \quad b = k \sin \alpha
\]

So that \(k = \sqrt{a^2 + b^2}\), \(\alpha = \tan^{-1}(b/a)\)

Thus,
\[
y_1 = \frac{dy}{dx} = e^{ax} [k(\cos \alpha) \sin(bx + c) + k(\sin \alpha) \cos(bx + c)]
\]
\[
= ke^{ax} \sin(bx + c + \alpha)
\]

Therefore,
\[
y_2 = \frac{d^2y}{dx^2} = k[ae^{ax} \sin(bx + c + \alpha) + be^{ax} \cos(bx + c + \alpha)]
\]
\[
= ke^{ax} [k(\cos \alpha) \sin(bx + c + \alpha) + k(\sin \alpha) \cos(bx + c + \alpha)]
\]
\[
= k^2 e^{ax} \sin(bx + c + 2\alpha)
\]
Proceeding like this, we obtain

\[ y_n = \frac{d^n y}{dx^n} = k^n e^{ax} \sin(bx + c + n\alpha) \]

Thus, we have the formula

\[ D^n[e^{ax} \sin(bx + c)] = (a^2 + b^2)^{n/2} e^{ax} \sin(bx + c + n \tan^{-1}(b/a)) \]

...(19)

In particular,

\[ D^n[e^{x} \sin x] = 2^{n/2} e^{x} \sin(x + n\pi/4) \]

...(20)

(7) \( n \)th derivative of \( e^{ax} \cos(bx + c) \)

Let

\[ y = e^{ax} \cos(bx + c) \]

Then

\[ y_1 = \frac{dy}{dx} = ae^{ax} \cos(bx + c) - be^{ax} \sin(bx + c) \]

\[ = e^{ax} [k(\cos\alpha)\cos(bx + c) - k(\sin\alpha)\sin(bx + c)] \]

\[ = ke^{ax} \cos(bx + c + \alpha) \]

Therefore,

\[ y_2 = \frac{d^2 y}{dx^2} = k[ae^{ax} \cos(bx + c + \alpha) - be^{ax} \sin(bx + c + \alpha)] \]

\[ = ke^{ax} [k(\cos\alpha)\cos(bx + c + \alpha) - k(\sin\alpha)\sin(bx + c + \alpha)] \]

\[ = k^2 e^{ax} \cos(bx + c + 2\alpha) \]

Proceeding like this, we obtain

\[ y_n = \frac{d^n y}{dx^n} = k^n e^{ax} \cos(bx + c + n\alpha) \]

Thus, we have the formula

\[ D^n[e^{ax} \cos(bx + c)] = (a^2 + b^2)^{n/2} e^{ax} \cos(bx + c + n \tan^{-1}(b/a)) \]

...(21)

In particular,

\[ D^n[e^{x} \cos x] = 2^{n/2} e^{x} \cos(x + n\pi/4) \]

...(22)

(8) \( n \)th derivative of \( a^{mx} \)

Let

\[ y = a^{mx} \]

Taking logarithm on both sides,

\[ \log y = mx \log a \]

Differentiating w.r.t. \( 'x' \), we get

\[ \frac{1}{y} \frac{dy}{dx} = m \log a \cdot 1 \]
\[ y_1 = \frac{dy}{dx} = (m \log a) \cdot y \]

\[ y_2 = \frac{d^2y}{dx^2} = m \log a \cdot y_1 \]

\[ = (m \log a)(m \log a) \cdot y \]

\[ = (m \log a)^2 y \]

\[ y_3 = \frac{d^3y}{dx^3} = (m \log a)^2 \cdot y_1 \]

\[ = (m \log a)^3 y \]

\[ \vdots \]

\[ y_n = \frac{d^n y}{dx^n} = (m \log a)^n \cdot y \]

\[ D^n[a^{mx}] = (m \log a)^n \cdot a^{mx} \]

**WORKED EXAMPLES**

**Example 1:** Find the \( n \)th derivative of the following functions:

\( i) \) \( \sin^3 x \) \quad \( ii) \) \( \cos^4 x \).

**Solution:**

\( i) \) \( y = \sin^3 x \)

where \( \sin^3 x = \frac{1}{4} (3\sin x - \sin 3x) \)

\[ D^n(\sin^3 x) = \frac{3}{4} D^n(\sin x) - \frac{1}{4} D^n(\sin 3x) \]

\[ = \frac{3}{4} \sin\left( x + \frac{n\pi}{2}\right) - \frac{3}{4} \sin\left( 3x + \frac{n\pi}{2}\right) \] \quad [Using the formula (15)]

\( ii) \) \( y = \cos^4 x \)

where \( \cos^2 x = \frac{1}{2} (1 + \cos 2x) \)

so that \( \cos^4 x = \frac{1}{4} (1 + \cos 2x)^2 \)

\[ = \frac{1}{4} [1 + \cos^2 2x + 2 \cos 2x] \]
where

\[ \cos^2 2x = \frac{1}{2} (1 + \cos 4x) \]

\[ \therefore \quad \cos^4 x = \frac{1}{4} \left[ 1 + \frac{1}{2} (1 + \cos 4x) + 2 \cos 2x \right] \]

\[ = \frac{1}{4} + \frac{1}{8} + \frac{1}{8} \cos 4x + \frac{1}{2} \cos 2x \]

\[ \cos^4 x = \frac{3}{8} + \frac{1}{2} \cos 2x + \frac{1}{8} \cos 4x \]

\[ \therefore \quad D^n (\cos^4 x) = D^n \left( \frac{3}{8} + \frac{1}{2} D^n (\cos 2x) + \frac{1}{8} D^n (\cos 4x) \right) \]

\[ = 0 + \frac{2^n}{2} \cos \left( 2x + \frac{n\pi}{2} \right) + \frac{4^n}{8} \cos \left( 4x + \frac{n\pi}{2} \right) \]

\[ = \frac{2^n}{2} \cos \left( 2x + \frac{n\pi}{2} \right) + \frac{4^n}{8} \cos \left( 4x + \frac{n\pi}{2} \right) \]

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**Example 2:** Find the \( n \)th derivative of the following:

(i) \( \sin h 2x \sin 4x \)

(ii) \( e^{-x} \sin^2 x \)

(iii) \( e^{2x} \cos^3 x \)

(iv) \( e^{-x} \sin x \cos 2x \)

(v) \( e^{ax} \cos^2 x \sin x \)

**Solution:**

(i) \( \sin h 2x \sin 4x = \frac{1}{2} (e^{2x} - e^{-2x}) \cdot \sin 4x \)

\[ \sin h 2x \sin 4x = \frac{1}{2} [e^{2x} \sin 4x - e^{-2x} \sin 4x] \]

\[ D^n (\sin h 2x \sin 4x) = \frac{1}{2} [D^n (e^{2x} \sin 4x) - D^n (e^{-2x} \sin 4x)] \]

\[ = \frac{1}{2} [2^n (e^{2x} \sin (4x + n \tan^{-1} 2) - e^{-2x} \sin (4x - n \tan^{-1} 2))] \]

Using the formula (19)

\[ \sin h 2x = \frac{e^{2x} - e^{-2x}}{2} \]
(ii) We have $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

Therefore,

$$D^n(e^{-x}\sin^2 x) = \frac{1}{2} D^n(e^{-x}) - \frac{1}{2} D^n(e^{-x} \cos 2x)$$

$$= \frac{1}{2} e^{-x}(-1)^n - \frac{1}{2} [5^{n/2} e^{-x} \cos(2x + n \tan^{-1}(-2))]$$

$$= \frac{1}{2} e^{-x}[(-1)^n - 5^{n/2} \cos(2x - n \tan^{-1}(2))]$$

(iii) We have $\cos^3 x = \frac{1}{4}(\cos 3x + 3 \cos x)$

$$D^n(e^{2x} \cos^3 x) = \frac{1}{4} D^n(e^{2x} \cos 3x) + \frac{3}{4} D^n(e^{2x} \cos x)$$

$$= \frac{1}{4} [(2^2 + 3^2)^{n/2} e^{2x} \cos(3x + n \tan^{-1}(3/2))]$$

$$+ \frac{3}{4} [(2^2 + 1^2)^{n/2} e^{2x} \cos(x + n \tan^{-1}(1/2))]$$

$$= \frac{1}{4} e^{2x} [13^{n/2} \cos(3x + n \tan^{-1}(3/2)) + 3(5)^{n/2} \cos(x + n \tan^{-1}(1/2))]$$

(iv) We have $\sin x \cos 2x = \frac{1}{2} (\sin 3x - \sin x)$

Therefore,

$$D^n(e^{-x} \sin x \cos 2x) = \frac{1}{2} D^n(e^{-x} \sin 3x) - \frac{1}{2} D^n(e^{-x} \sin x)$$

$$= \frac{1}{2} [((-1)^2 + 3^2)^{n/2} e^{-x} \sin(3x + n \tan^{-1}(-3))]$$

$$- \frac{1}{2} [((-1)^2 + 1^2)^{n/2} e^{-x} \sin(x + n \tan^{-1}(-1))]$$

$$= \frac{1}{2} e^{-x}[10^{n/2} \sin(3x - n \tan^{-1}3) - 2^{n/2} \sin(x - n\pi/4)]$$

(v) We note that

$$\cos^2 x \sin x = \frac{1}{2}(1 + \cos 2x) \cdot \sin x$$
\[ \sin x + \frac{1}{2} \sin 3x - \frac{1}{4} \]