CHAPTER 4

DIRECT NUMERICAL INTEGRATION METHODS

4.1 INTRODUCTION

The behavior of many dynamic systems undergoing time-dependant changes (transients) can be described by ordinary differential equations. When the solution to the differential equation(*s*) of motion of a dynamic system cannot be obtained in closed form, a numerical procedure is warranted. Many numerical integration methods are available for the approximate solution of such equation(*s*) of motion. All the numerical integration methods have two basic characteristics. First, they do not satisfy the differential equation(*s*) at all time *t*, but only at discrete time intervals, say Δt apart. Secondly, within each time interval Δt, a specific type of variation of the displacement *X*, velocity $\dot{X}$, and acceleration $\ddot{X}$ is assumed. Thus several numerical integration schemes are available depending on the type of variation assumed for $X, \dot{X}$ and $\ddot{X}$ within each time interval Δt.

In this chapter we discuss several widely used step-by-step numerical integration schemes for solutions of both single and multi degree of freedom systems. A brief description of these methods is presented for linear dynamic response analysis and their application is illustrated by several examples.

4.2 SINGLE DEGREE OF FREEDOM SYSTEM

The general equation of a viscously damped single degree of freedom dynamical system, which is linear, can be expressed in the following general form:

$$M\ddot{X} + C\dot{X} + KX = F(t) \tag{4.1}$$

where *M*, *C* and *K* are the mass, damping and stiffness of the system; *F* (*t*) is the applied force; and X, $\dot{X}$ and $\ddot{X}$ are the displacement, velocity and acceleration of the system.

4.2.1 Finite Difference Method

If the equilibrium relation (4.1) is regarded as an ordinary differential equation with constant coefficients, it follows that any convenient *finite difference* expressions to approximate the velocities and accelerations in terms of displacements can be used. The central idea in the *finite difference* method is to use approximations to derivatives. Hence, the general differential equation such as (4.1) and the associated boundary conditions, if any, are replaced by the corresponding finite difference equations. The continuous variable *t* is replaced by the discrete variable t_i and the differential equation is solved progressively in time increments $h = \Delta t$ starting from known initial conditions. The solution obtained is approximate but by suitably selecting the time increment the accuracy of the solution can be improved.

In this method, we replace the solution domain with a finite number of points, known as mesh or grid points, and obtain the solution at these points. The mesh or grid points are generally equally spaced along the independent coordinate as shown in Fig. 4.1. Central difference method is based on the Taylor's series expansion of X_{i+1} and X_{i-1} about the grid point i.

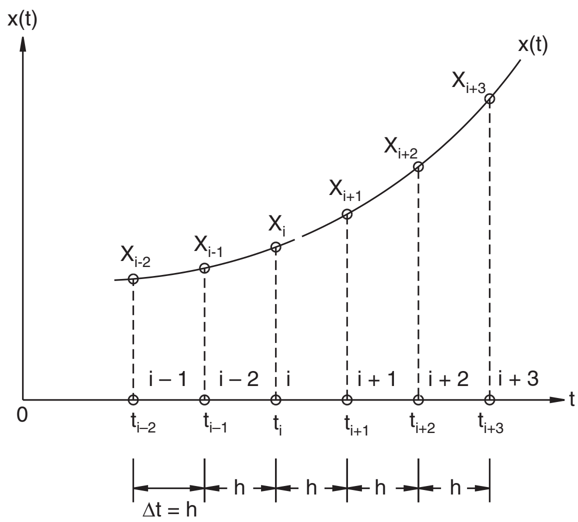

Fig. 4.1

$$X_{i+1} = X_i + h\dot{X}_i + \frac{h_2}{2}\ddot{X}_i + \frac{h^3}{6}\dddot{X}_i + \ldots \tag{4.2}$$

$$X_{i-1} = X_i - h\dot{X}_i + \frac{h_2}{2}\ddot{X}_i - \frac{h^3}{6}\dddot{X}_i + \ldots \tag{4.3}$$

where $X_i = X(t = t_i)$ and the interval $h = t_{i+1} - t_i = \Delta t$. By taking the first two terms only and subtracting Eq. (4.3) from (4.2), we obtain

$$\dot{X}_i = \frac{1}{2h}(X_{i+1} - X_{i-1}) \tag{4.4}$$

Adding Eqs. (4.3) and (4.2), we get

$$\ddot{X}_i = \frac{1}{h^2}(X_{i-1} - 2X_i + X_{i+1}) \tag{4.5}$$

Although there exist a number of finite difference schemes, here we consider only two methods selected for their simplicity. They are the central difference method and the Runge-Kutta method.

4.2.2 Central Difference Method

Let the duration over which the numerical solution of Eq. (4.1) is required be divided into n equal parts of interval $h = \Delta t$ each. The initial conditions are assumed to be given by

$$X(t = 0) = X_o \text{ and } \dot{X}(t = 0) = \dot{X}_0.$$

The accuracy of the solution always depends on the size of the time step. The critical time step is given by $\Delta t_{cri} = \tau_n / \pi$, where τ_n is the natural period of the system. If Δt is selected to be larger than Δt_{cri}, the method becomes unstable, that is the truncation of higher order terms or rounding-off in the computer causes errors to grow and makes the dynamic response calculations meaningless. Numerical methods, which require the use of time, step Δt smaller than the critical time step Δt_{cri} are said to be conditionally stable. A safe rule to use is to choose $h \le \tau_n / 10$. Substituting Eqs. (4.5) and (4.4) for $\ddot{X}_i$ and $\dot{X}_i$ respectively in Eq. (4.1) at mesh or grid point i give

$$M\left\{\frac{X_{i+1} - 2X_i + X_{i-1}}{(\Delta t)^2}\right\} + C\left\{\frac{X_{i+1} - X_{i-1}}{2\Delta t}\right\} + KX_i = F_i \tag{4.6}$$

where $X_i = X(t_i)$ and $F_i = F(t_i)$. Solving Eq. (4.6) for X_{i+1} gives

$$X_{i+1} = \left\{\frac{1}{\dfrac{M}{(\Delta t)^2} + \dfrac{C}{2\Delta t}}\right\}\left[\left\{\frac{2M}{(\Delta t)^2} - K\right\}\{X_i\} + \left\{\frac{C}{2\Delta t} - \frac{M}{(\Delta t)^2}\right\}X_{i-1} + F_i\right] \tag{4.7}$$

which is known as the recurrence formula.

Thus the displacement of the mass, X_{i+1}, can be calculated using Eq. (4.7) if we know the previous displacements, X_i, X_{i-1} and the present external force F_i. Repeated application of Eq. (4.7) gives us the response time history of the behavior of the system. Since the solution of X_{i+1} given in Eq. (4.7) is based on the previous displacement X_i given in Eq. (4.6), the integration procedure is known as an explicit integration method. Note that in order to compute X_1, both X_o and X_{-1} are required but the initial conditions provide only the values of X_o and $\dot{X}_o$. Therefore the central difference method is not self-starting. However, the value of X_{-1} can be obtained from Eqs. (4.4) and (4.5) as follows. First calculate the value of $\ddot{X}_o$ by substituting the given values of X_o and $\dot{X}_o$ in Eq. (4.1) as follows,

$$\ddot{X}_o = \frac{1}{M}[F(t = 0) - C\dot{X}_o - KX_o] \tag{4.8}$$

The value of X_{-1} is then obtained by the application of Eqs. (4.4) and (4.5) at $i = 0$.

$$X_{-1} = X_o - \Delta t\,\dot{X}_o + \frac{(\Delta t)^2}{2}\ddot{X}_o \tag{4.9}$$

4.2.3 Runge-Kutta Method

In the Runge-Kutta method, an approximation to $X_{(t+\Delta t)}$ is obtained from X_t in such a way that the power series expansion of the approximation coincides, up to terms of a certain order $(\Delta t)^N$ in the time interval Δt, with the actual Taylor series expansion of $(t + \Delta t)$ in powers of Δt. The method is based on the assumption that the higher derivatives exist at points required.

The Runge-Kutta method is self-starting and has the advantage that no initial values are needed beyond the prescribed values. A brief discussion of its basis is represented here. In the Runge-Kutta method the second-order differential equation is first reduced to two first-order equations. Consider the differential equation for the single degree of freedom system given in Eq. (4.1). Equation (4.1) can be rewritten as

$$\ddot{X} = \frac{1}{M}[F(t) - C\dot{X} - KX] = f(X, \dot{X}, t) \tag{4.10}$$

By letting $X_1 = X$ and $X_2 = \dot{X}$, Eq. (4.10) can be reduced to the following two first-order equations:

$$\begin{aligned} \dot{X}_1 &= X_2 \\ \dot{X}_2 &= f(X_{1,}\ X_2, t) \end{aligned} \tag{4.11}$$

By defining

$$\overline{X}(t) = \begin{Bmatrix} X_1(t) \\ X_2(t) \end{Bmatrix}$$

and

$$\overline{F}(t) = \begin{Bmatrix} X_2 \\ f(X_1, X_2, t) \end{Bmatrix}$$

the following recurrence formula is obtained to find the values of $\overline{X}(t)$ at mesh or grids points t_i according to the fourth order Runge-Kutta method. We omit the details of the derivation of the method.

$$\overline{X}_{i+1} = \overline{X}_i + \frac{1}{6}[\overline{K}_1 + 2\overline{K}_2 + 2\overline{K}_3 + \overline{K}_4]$$

where $\overline{K}_1 = h\overline{F}(\overline{X}_i, t_i)$

$$\overline{K}_2 = h\overline{F}(\overline{X}_i + \frac{1}{2}\overline{K}_1, t_i + \frac{1}{2}h)$$

$$\overline{K}_3 = h\overline{F}(\overline{X}_i + \frac{1}{2}K_2, t_i + \frac{1}{2}h) \tag{4.12}$$

and

$$\overline{K}_4 = h\overline{F}(\overline{X}_i + \frac{1}{2}\overline{K}_3, t_{i+1})$$

Although the Runge-Kutta method does not require the computation of derivatives beyond the first, its higher accuracy is obtained by four evaluations of the first derivatives to obtain agreement with the Taylor series solution through terms of order h^4. Since the fourth order Runge-Kutta method is an explicit method, the maximum time step is usually governed by stability considerations. The change in time step can be easily implemented between iterations and hence the method can be considered as an inherently stable method. The main drawback of the method is that each forward step requires several computations of the functions thus increasing the computational cost. The Runge-Kutta method is applicable and extendable to a system of differential equations.

4.3 MULTI DEGREE OF FREEDOM SYSTEM

The general form of the equations of motion for a multi-degree of freedom system are written as

$$[M]\{\ddot{X}\} + [C]\{\dot{X}\} + [K]\{X\} = \{F(t)\} \tag{4.13}$$

where $[M], [C], \text{and } [K]$ are the mass, damping and stiffness matrices for the system and $[\ddot{X}], [\dot{X}], \text{and } [X]$ refer to the acceleration, velocity, and displacement vectors, respectively. $\{F(t)\}$ is the force vector. Several numerical direct integrating schemes are available to determine the approximate solution of a system of equations of motion. For a linear dynamic system, matrices $[M], [C] \text{ and } [K]$ are independent of time and therefore remain unchanged during the integration procedure. These matrices vary with time for a nonlinear dynamic system and must be modified during the integration of equations of motion. For the solution of equations of motion for a linear dynamic system, either the normal mode superposition method of dynamic analysis or direct numerical integration methods can be used. However, for the solution of nonlinear equations of motion, direct numerical integration methods are generally recommended.

In a direct integration method, the system of equations of motion is integrated successively by using a step-by-step numerical procedure. No transformation of the equations of motion is needed prior to integration and using difference formulas that involve one or more increments of time usually approximates time derivatives. Basically there are two principal approaches used in the direct integration method: explicit and implicit schemes. In an explicit scheme, the response quantities are expressed in terms of previously determined values of displacement, velocity and acceleration. In an implicit scheme, the difference equations are combined with the equations of motion, and the displacements are calculated directly by solving the equations.

In this section, only selected numerical integration schemes widely used for linear and nonlinear dynamic analyses are considered. Three explicit and four implicit direct integration schemes are examined. A brief description of these schemes is presented and their application is illustrated. The explicit schemes presented are the central difference method, two-cycle iteration with trapezoidal rule, and fourth-order Runge-Kutta. The implicit schemes include the Houbolt, Wilson-Theta, Newark-Beta, and Park Stiffly stable methods. The accuracy, stability, and efficiency of these schemes are examined by comparing the results for sample problems.

4.4 EXPLICIT SCHEMES

As mentioned earlier, in an explicit formulation, the response quantities are expressed in terms of previously determined values of displacement, velocity and acceleration.

4.4.1 Central Difference Method

The procedure indicated for the case of a single degree of freedom system can be directly extended to this case. Consider a displacement time history curve as shown in Fig. 4.2. At the middle of the time interval Δt, the velocity is given by

$$\dot{X}_{i+\frac{1}{2}} = \frac{X_{i+1} - X_i}{\Delta t} \tag{4.14}$$

and

$$\dot{X}_{i-\frac{1}{2}} = \frac{X_i - X_{i-1}}{\Delta t} \tag{4.15}$$

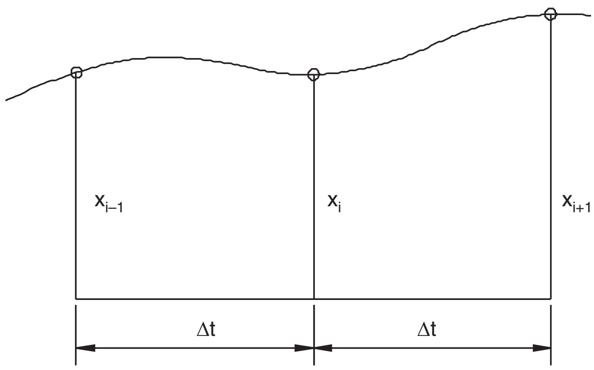

Fig 4.2

The acceleration is

$$\ddot{X}_i = \frac{\dot{X}_{i+\frac{1}{2}} - \dot{X}_{i-\frac{1}{2}}}{\Delta t} \tag{4.16}$$

Substituting $\dot{X}_{i+\frac{1}{2}}$ and $\dot{X}_{i-\frac{1}{2}}$ from the Eqs. (4.14) and (4.15) into Eq. (4.16), we get

$$\ddot{X}_i = \frac{1}{\Delta t^2}\,(X_{i+1} - 2X_i + X_{i-1}) \tag{4.17}$$

The difference formulas in the central difference method for velocity and acceleration are written in terms of displacement as

$$\{\dot{X}_t\} = \frac{1}{2\Delta t}\,[\{X_{t+\Delta t}\} - \{X_{t-\Delta t}\}] \tag{4.18}$$

$$\{\ddot{X}_t\} = \frac{t}{\Delta t^2}\,[\{X_{t+\Delta t}\} - 2\{X_t\} + \{X_{t-\Delta t}\}] \tag{4.19}$$

Substituting $\{\dot{X}_t\}$ and $\{\ddot{X}_t\}$ from Eqs. (4.18) and (4.19), respectively into Eq. (4.13), we get

$$[\overline{M}]\,\{X_{t+\Delta t}\} = \{\overline{F}_t\} \tag{4.20}$$

where $[\overline{M}]$, the effective mass matrix, and $[\overline{F}_t]$, the effective force vector, is given by

$$[\overline{M}] = \frac{1}{\Delta t^2}\,[M]\,\frac{1}{2\Delta t}\,[C] \tag{4.21}$$

$$[\overline{F}_t] = \{F_t\} - ([K] - \frac{2}{\Delta t^2}\,[M])\,\{X_t\} - (\frac{1}{\Delta t^2}\,[M] - \frac{1}{2\Delta t}\,[C])\,\{X_{t-\Delta t}\} \tag{4.22}$$

At time $t + \Delta t$, the displacements $\{X_{t+\Delta t}\}$ can be computed by solving Eq. (4.20), and the velocities and accelerations at time t are determined by substituting these values of $\{X_{t+\Delta t}\}$ into Eqs. (4.18) and (4.19). Note that the calculation of $\{X_{t+\Delta t}\}$ involves $\{X_t\}$ and $\{X_{t-\Delta t}\}$. Hence, in order to obtain the solution at time Δt, a special starting procedure is needed. Table 4.1 summarizes the time integration schedule as suitable for integration in the computer.

Table 4.1

Algorithms based on the central-difference method

(*a*) Initial computations: 1. Form stiffness [*K*], mass [*M*], and damping [*C*] matrices. 2. Initialize $\{X_0\}$, $\{\dot{X}_0\}$, and $\{\ddot{X}_0\}$. 3. Select time step Δt and calculate integration constants a_i: $a_0 = \frac{1}{\Delta t^2}; a_1 = \frac{1}{2\Delta t}; a_2 = 2a_0; a_3 = \frac{1}{a_2}.$ 4. Calculate $\{X_{-\Delta t}\} = \{X_0\} - \Delta t\,\{\dot{X}_0\} + a_3\,\{\ddot{X}_0\}$. 5. Form effective mass matrix $\lfloor \overline{M} \rfloor = a_0\,[M] + a_1\,[C]$. 6. Triangularize $[\overline{M}] : [\overline{M}] = [L]\,[D]\,[L]^T$
(*b*) For each time step: 1. Calculate effective force vector at time t: $\{\overline{F}\} = \{F_t\} - ([K] - a_2\,[M])\,\{X_t\} - (a_0\,[M] - a_1\,[C])\,\{X_{t-\Delta t}\}$ 2. Solve for displacements at time $t + \Delta t$: $\lfloor \overline{M} \rfloor \{X_{t+\Delta t}\} = \lfloor \overline{F_t} \rfloor$ 3. Calculate $\{\dot{X}\}$ and $\{\ddot{X}\}$ at time t: $\{\dot{X}_t\} = a_1(-\{X_{t-\Delta t}\} - \{X_{t+\Delta t}\})$ $\{\ddot{X}_t\} = a_0\,(\{X_{t-\Delta t}\} - 2\{X_t\} + \{X_{t+\Delta t}\})$

The local truncation error of this method is of the order of Δt^2. An important consideration in the use of the central difference method is that the integration method requires that the time step Δt smaller than a critical value, Δt_{cr}, which is limited by the highest frequency of the discrete system ω_{max}, where

$$\Delta t \le \Delta t_{cr} \le \frac{2}{\omega_{max}} \qquad ...(4.23)$$

If the time step is longer than Δt_{cr}, the integration is unstable, meaning that any errors resulting from the numerical integration of round off in the computer grow and make the dynamic response calculations questionable.

4.4.2 Two-Cycle Iteration with Trapezoidal Rule

The equations of motion at any time t are expressed in the incremental form as

$$[M]\,\{\Delta\ddot{X}_t\} = \{\Delta F_t\} - [K]\,\{\Delta X_t\} - [C]\,\{\Delta\dot{X}_t\} \qquad (4.24)$$

The increments in the velocities and displacements are estimated by the use of the following relationships in the first iteration cycle:

For the first time step:

$$\{\Delta \dot{X}_t\} = \Delta t \,\{\ddot{X}_{t-\Delta t}\} \tag{4.25}$$

For other time steps:

$$\{\Delta \dot{X}_t\} = 2\Delta t \,\{\ddot{X}_{t-\Delta t}\} - \{\Delta \dot{X}_{t-\Delta t}\} \tag{4.26}$$

$$\{\dot{X}_t\} = \{\dot{X}_{t-\Delta t}\} + \{\Delta \dot{X}_t\} \tag{4.27}$$

$$\{\Delta X_t\} = \frac{1}{2}\Delta t \,[\{\dot{X}_{t-\Delta t}\} + \{\dot{X}_t\}] \tag{4.28}$$

By substituting the relations $\{\Delta \dot{X}_t\}$ and $\{\Delta X_t\}$ from Eqs. (4.26) and (4.28) into Eq. (4.24) we obtain the increments in the accelerations as

$$\{\Delta \ddot{X}_t\} = [M]^{-1}\,(\{\Delta F_t\} - [K]\,\{\Delta X_t\} - [C]\,\{\Delta \dot{X}_t\}) \tag{4.29}$$

These are then employed to obtain the estimate of the acceleration at time t as

$$\{\ddot{X}_t\} = \{\ddot{X}_{t-\Delta t}\} + \{\Delta \ddot{X}_t\} \tag{4.30}$$

In the second iteration cycle, the increments in the velocities and the accelerations are refined as

$$\{\Delta \dot{X}_t\} = \frac{1}{2}\Delta t(\{\ddot{X}_{t-\Delta t}\} + \{\ddot{X}_t\}) \tag{4.31}$$

$$\{\dot{X}_t\} = \{\dot{X}_{t-\Delta t}\} + \{\Delta \dot{X}_t\} \tag{4.32}$$

$$\{\Delta X_t\} = \frac{1}{2}\Delta t(\{\dot{X}_{t-\Delta t}\} + \{\dot{X}_t\}) \tag{4.33}$$

Finally, the relations for $\{\Delta \dot{X}_t\}$ and $\{\Delta X_t\}$ in Eqs. (4.31) and (4.33) are substituted into Eq. (4.29) to compute the new increments in the accelerations. These are then used in Eq. (4.30) to calculate the accelerations at time t. The computational algorithm based on this method is summarized in Table 4.2.

Table 4.2

Algorithm based on two-cycle iteration with trapezoidal rule

1. Form stiffness [K], mass [M] and damping [C] matrices
2. Initialize $\{X_0\}, \{\dot{X}_0\}$ and $\{\ddot{X}_0\}$
3. Select time step Δt and calculate for the first time step in the first iteration cycle: $\{\Delta \dot{X}_t\} = \Delta t\,\{\ddot{X}_{t-\Delta t}\}$
4. For other time steps: $\{\Delta \dot{X}_t\} = 2\Delta t\,\{\ddot{X}_{t-\Delta t}\} - \{\Delta \dot{X}_{t-\Delta t}\}$ $\{\dot{X}_t\} = \{\dot{X}_{t-\Delta t}\} + \{\Delta \dot{X}_t\}$ $\{\Delta X_t\} = \frac{\Delta t}{2}\,(\{\dot{X}_{t-\Delta t}\} - \{\dot{X}_t\})$

5. Compute $\{\Delta\ddot{X}_t\}$ and $\{\dot{X}_t\}$ at time t: $\{\Delta\ddot{X}_t\} = [M]^{-1}(\{\Delta F_t\} - [K]\{\Delta X_t\} - [C]\{\Delta\dot{X}_t\})$ $\{\ddot{X}_t\} = \{\ddot{X}_{t-\Delta t}\} + \{\Delta\ddot{X}_t\}$
6. Second iteration cycle: $\{\Delta\dot{X}_t\} = \frac{\Delta t}{2}(\{\ddot{X}_{t-\Delta t}\} + \{\ddot{X}_t\})$ $\{\dot{X}_t\} = \{\dot{X}_{t-\Delta t}\} + \{\Delta\dot{X}_t\}$ $\{\Delta X_t\} = \frac{\Delta t}{2}(\{\dot{X}_{t-\Delta t}\} + \{\dot{X}_t\})$
7. Compute $\{\Delta\ddot{X}_t\}$ in step 5 using $\{\Delta\dot{X}_t\}, \{\dot{X}_t\}$ and $\{\Delta X_t\}$ from step 6.
8. Finally, compute $\{\ddot{X}_t\}$ from step 5, using $\{\ddot{X}_t\}$ and $\{\Delta\ddot{X}_t\}$ from step 7.

4.4.3 Fourth Order Runge-Kutta Method

In the fourth order Runge-Kutta method, the system of second order differential Eq. (4.13) are converted into state variable form. That is, both the displacements and velocities are treated as unknowns $\{y\}$ defined by

$$\{y\} = \begin{Bmatrix} \{X\} \\ \{\dot{X}\} \end{Bmatrix} \tag{4.34}$$

Using Eq. (4.33), Eq.(4.13) can be rewritten as

$$\{\ddot{X}\} = -[M]^{-1}[K]\{X\} - [M]^{-1}[C]\{\dot{X}\} + [m]^{-1}\{F(t)\} \tag{4.35}$$

Using the identity

$$\{\dot{y}\} = \{\dot{y}\} \tag{4.36}$$

we obtain from Eqs. (4.35) and (4.36)

$$\{\dot{y}\} = \begin{Bmatrix} \{\dot{X}\} \\ \{\ddot{X}\} \end{Bmatrix} = \begin{bmatrix} [0] \\ -[M]^{-1}[K] \end{bmatrix} \begin{bmatrix} [I] \\ -[M]^{-1}[C] \end{bmatrix} \begin{Bmatrix} \{X\} \\ \{\dot{X}\} \end{Bmatrix} + \begin{Bmatrix} 0 \\ [M]^{-1}\{F(t)\} \end{Bmatrix} \tag{4.37}$$

or

$$\{\dot{y}\} = [E]\{y\} + \{F^*(t)\} \tag{4.38}$$

That is

$$\{\dot{y}\} = \{f(t, y)\} \tag{4.39}$$

In the Runge-Kutta method, an approximation to $\{y_{y+\Delta t}\}$ is obtained from $\{y_t\}$ in such a way that the power series expansion of the approximation coincides, upto the terms of a certain order $(\Delta t)^N$ in the time interval Δt, with the actual Taylor series expansion of $(t + \Delta t)$ in powers of Δt. This method has the advantage that no initial values are required beyond the prescribed ones. The general fourth order algorithms are based on formulas of the form

$$\{y_{t+\Delta t}\} = \{y_t\} + \Delta t(aK_1 + bK_2 + cK_3 + dK_4) \tag{4.40}$$

where a, b, c and d are constants and K_1, K_2, K_3, and K_4, are the approximate derivative values computed in the interval $t_K < t < t_{K+\Delta t}$. Several fourth order algorithms have been proposed. The following is due to Runge-Kutta and we omit presenting the details of its derivation.

$$\{y_{t+\Delta t}\} = \{y_t\} + \frac{\Delta t}{6}\,[K_1 + 2K_2 + 2K_3 + K_4] \tag{4.41}$$

in which

$$K_1 = f(t, y_t)$$
$$K_2 = f\left(t + \frac{\Delta t}{2}, y_t + K_1\frac{\Delta t}{2}\right)$$
$$K_3 = f\left(t + \frac{\Delta t}{2}, y_t + K_2\frac{\Delta t}{2}\right)$$
$$K_4 = f(t + \Delta t, y_t + K_3\Delta t) \tag{4.42}$$

The Runge-Kutta algorithm does not require the calculation of higher derivatives. This method is completely self-starting and the step size can be changed easily between iterations and hence the method can be considered inherently stable. The truncation error e_t for the fourth order Runge-Kutta scheme is of the form

$$e_t = c(\Delta t)^5 \tag{4.43}$$

where c is constant, which depends on $f(t, y)$ and its higher-order partial derivatives. Runge-Kutta method generates an artificial damping that unduly suppresses the amplitude of the response of a dynamic system to some extent.

4.5 IMPLICIT SCHEMES

In an implicit scheme, the difference equations are combined with the equations of motion, and the displacements are calculated directly by solving the equations.

4.5.1 Houbolt Method

The Houbolt method is based on third-order interpolation of displacements X_t, and the multistep implicit formulas for $\dot{X}_t$ and $\ddot{X}_t$ are obtained in terms of X_t by using backward differences.

The difference formulas are summarized in the following with reference to Fig. 4.3. By considering a cubic curve that passes through the four successive ordinates, the following equations can be obtained:

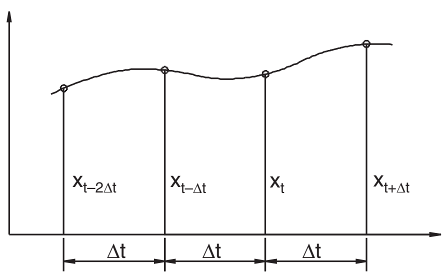

Fig. 4.3

$$X_t = X_{t-\Delta t} - \Delta t\dot{X}_{t+\Delta t} + \frac{\Delta t^2}{2}\ddot{X}_{t+\Delta t} - \frac{\Delta t^3}{6}\dddot{X}_{t+\Delta t} \tag{4.44}$$

$$X_{t-\Delta t} = X_{t+\Delta t} - (2\Delta t)\dot{X}_{t+\Delta t} + \frac{(2\Delta t)^2}{2}\ddot{X}_{t+\Delta t} - \frac{(2\Delta t)^3}{6}\dddot{X}_{t+\Delta t} \tag{4.45}$$

$$X_{t-2\Delta t} = X_{t+\Delta t} - (3\Delta t)\dot{X}_{t+\Delta t} + \frac{(3\Delta t)^2}{2}\ddot{X}_{t+\Delta t} - \frac{(3\Delta t)^3}{6}\dddot{X}_{t+\Delta t} \tag{4.46}$$

Solving Eqs. (4.44) to (4.46) for $\ddot{X}_{t+\Delta t}$ and $\dot{X}_{t+\Delta t}$ we get

$$\ddot{X}_{t+\Delta t} = \frac{1}{\Delta t^2}(2X_{t+\Delta t} - 5X_t + 4X_{t-\Delta t} - X_{t-2\Delta t}) \tag{4.47}$$

$$\dot{X}_{t+\Delta t} = \frac{1}{6\Delta t}(11X_{t+\Delta t} - 18X_t + 9X_{t-\Delta t} - 2X_{t-2\Delta t}) \tag{4.48}$$

The difference formulas in the Houbolt algorithm are therefore given by

$$\{\ddot{X}_{t+\Delta t}\} = \frac{1}{\Delta t^2}[2\{X_{t+\Delta t}\} - 5\{X_t\} + 4\{X_{t-\Delta t}\} - \{X_{t-2\Delta t}\}] \tag{4.49}$$

$$\{\dot{X}_{t+\Delta t}\} = \frac{1}{6\Delta t}[11\{X_{t+\Delta t}\} - 18\{X_t\} + 9\{X_{t-\Delta t}\} - 2\{X_{t-2\Delta t}\}] \tag{4.50}$$

By substituting the expressions for $\{\ddot{X}_{t+\Delta t}\}$ and $\{\dot{X}_{t+\Delta t}\}$ from (4.49) and (4.50), respectively, into (4.13), we get

$$\lfloor \overline{M} \rfloor \{X_{t+\Delta t}\} = \{\overline{F}_{t+\Delta t}\} \tag{4.51}$$

where $\lfloor \overline{M} \rfloor$ is the effective mass matrix and $\{\overline{F}_{t+\Delta t}\}$ is the effective force vector.

$$[\overline{M}] = \frac{2}{\Delta t^2}[M] + \frac{11}{6\Delta t}[C] + [K] \tag{4.52}$$

$$\{\overline{F}_{t+\Delta t}\} = \{F_{t+\Delta t}\} + (\frac{5}{\Delta t^2}[M] + \frac{3}{\Delta t}[C])\{X_t\}$$
$$-(\frac{4}{\Delta t^2}[M] + \frac{3}{2\Delta t}[C])\{X_{t-\Delta t}\} + (\frac{1}{\Delta t^2}[M] + \frac{1}{3\Delta t}[C])\{X_{t-2\Delta t}\} \tag{4.53}$$

Note that the equilibrium equation at time $t + \Delta t$, Eq. (4.51) is used in finding the solution for $\{X_{t+\Delta t}\}$. For this reason, this method is called an implicit integration method. It can be seen that the velocities and accelerations at time $t + \Delta t$ are obtained by substituting for $\{X_{t+\Delta t}\}$ in (4.50) and (4.49) respectively. Also that a knowledge of X_t, $X_{t-\Delta t}$, and $X_{t-2\Delta t}$ is needed to find solution for $\{X_{t+\Delta t}\}$. Since there is no direct method available to find $\{X_{t-\Delta t}\}$ and $\{X_{t-2\Delta t}\}$, initially we use the central difference method to find solution at time Δt and $2\Delta t$. This makes the method non-self-starting. The method also requires large computer storage to store displacements for the previous time steps. The step-by-step procedure to be used in the Houbolt method is summarized in Table 4.3. A basic difference between the Houbolt method in Table 4.3 and the central difference method in Table 4.2 is the appearance of the stiffness matrix K as a factor to the required displacements $X_{t+\Delta t}$. The term $KX_{t+\Delta t}$ appears because the equilibrium is considered at time $t + \Delta t$ and not at time t as in the central-difference method. There is no critical time-step limit, and Δt can in general be selected much larger than that given for the central-difference method.

Table 4.3

Algorithm based on Houbolt Method

(*a*) Initial Computations:

1. Form stiffness [*K*], mass [*M*] and damping [*C*] matrices
2. Initialize $\{X_0\}, \{\dot{X}_0\}$ and $\{\ddot{X}_0\}$.
3. Select time step Δt and calculate integration constants:

$$a_0 = \frac{2}{\Delta t^2}; a_1 = \frac{11}{6\Delta t}; a_2 = \frac{5}{\Delta t^2}; a_3 = \frac{3}{\Delta t}; a_4 = -2a_0; a_5 = \frac{-a_3}{2};$$

$$a_6 = \frac{a_0}{2}; a_7 = \frac{a_3}{9}.$$

4. Use special starting procedure, such as central-difference method to calculate $\{X_{\Delta t}\}$ and $\{X_{2\Delta t}\}$.
5. Calculate effective stiffness matrix:

$$\lfloor \overline{K} \rfloor = [K] + a_0\,[M] + a_1\,[C]$$

6. Triangularize $[\overline{K}]: [\overline{K}] = [L]\,[D]\,[L]^T$

(*b*) For each time step:

1. Calculate effective force vector at time $t + \Delta t$:

$$\{\overline{F}_{t+\Delta t}\} = \{F_{t+\Delta t}\} + [M]\,(a_2\,\{X_t\} + a_4\,\{X_{t-\Delta t}\} + a_6\,\{X_{t-2\Delta t}\})$$
$$+[C]\,(a_3\{X_t\} + a_5\,\{X_{t-\Delta t}\} + a_7\,\{X_{t-2\Delta t}\})$$

2. Solve for displacements at time $t + \Delta t$

$$\lfloor \overline{K} \rfloor \{X_{t+\Delta t}\} = \{\overline{F_{t+\Delta t}}\}$$

3. Calculate $\{\dot{X}\}$ and $\{\ddot{X}\}$ at time $t + \Delta t$:

$$\{\dot{X}_{t+\Delta t}\} = a_1\,\{X_{t+\Delta t}\} - a_3\,\{X_t\} - a_5\,\{X_{t-\Delta t}\} - a_7\,\{X_{t-2\Delta t}\}$$

$$\{\dot{X}_{t+\Delta t}\} = a_0\,\{X_{t+\Delta t}\} - a_2\,\{X_t\} - a_4\,\{X_{t-\Delta t}\} - a_6\,\{X_{t-2\Delta t}\}$$

4.5.2 Wilson Theta Method

The Wilson Theta method assumes that the acceleration of the system varies linearly between two instants of time. Referring to Fig. 4.4, the acceleration is assumed to be linear from time t between $t_i = i\Delta t$ to time $t_{i+\theta} = t_i + \theta\Delta t$, where $\theta \geq 1.0$. Because of this reason, the method is known as the Wilson Theta method.

If τ is the increase in time t between t and $t + \theta\Delta t$ $(0 \leq \tau \leq \theta\Delta t)$, then for time interval t to $t + \theta\Delta t$, it can be assumed that

$$\ddot{X}_{t+\tau} = \ddot{X}_t + \frac{\tau}{\theta\Delta t}\,(\ddot{X}_{t+\theta\Delta t} - \ddot{X}_t) \tag{4.54}$$

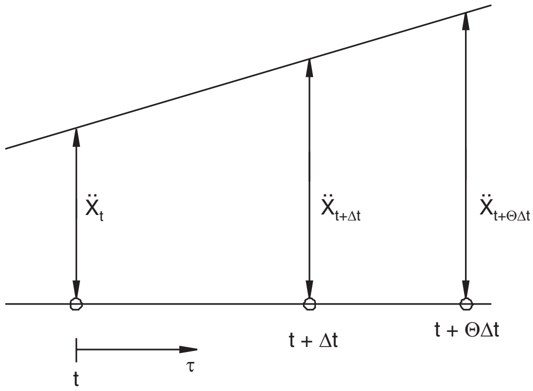

Fig. 4.4

Successive integration for Eq. (4.54) gives the following expressions for $\ddot{X}_{t+\tau}$ and $X_{t+\tau}$:

$$\dot{X}_{t+\tau} = \dot{X}_t + \ddot{X}_t \, \tau + \frac{\tau^2}{2\theta\Delta t} (\ddot{X}_{t+\theta\Delta t} - \ddot{X}_t) \tag{4.55}$$

$$X_{t+\tau} = X_t + \dot{X}_t \, \tau + \frac{1}{2} \ddot{X}_t \, \tau^2 + \frac{\tau^3}{6\theta\Delta t} (\ddot{X}_{t+\theta\Delta t} - \ddot{X}_t) \tag{4.56}$$

Substituting $\tau = \theta\Delta t$ into the above Eqs. (4.55) and (4.56), we obtain the following expressions for $\dot{X}$ and X at time $t + \theta\Delta t$:

$$\dot{X}_{t+\theta\Delta t} = \dot{X}_t + \frac{\theta\Delta t}{2} (\ddot{X}_t + \ddot{X}_{t+\theta\Delta t}) \tag{4.57}$$

$$X_{t+\theta\Delta t} = X_t + \theta\Delta t \, \dot{X}_t + \frac{\theta^2 \Delta t^2}{6} (\ddot{X}_{t+\theta\Delta t} + 2\ddot{X}_t) \tag{4.58}$$

Solving Eqs. (4.57) and (4.58) for $\ddot{X}_{t+\theta\Delta t}$ and $\dot{X}_{t+\theta\Delta t}$ in terms of $X_{t+\theta\Delta t}$, we get

$$\ddot{X}_{t+\theta\Delta t} = \frac{6}{\theta^2 \Delta t^2} (X_{t+\theta\Delta t} - X_t) - \frac{6}{\theta\Delta t} (\dot{X}_t) - 2\,(\ddot{X}_t)$$

$$\dot{X}_{t+\theta\Delta t} = \frac{3}{\theta\Delta t} (X_{t+\theta\Delta t} - X_t) - 2\dot{X}_t - \frac{\theta\Delta t}{2} \ddot{X}_t \tag{4.59}$$

The difference formulas in the Wilson Theta algorithm are then given by

$$\{\ddot{X}_{t+\theta\Delta t}\} = \frac{6}{\theta^2 \Delta t^2} (\{X_{t+\theta\Delta t}\} - \{X_t\}) - \frac{6}{\theta\Delta t} \{\dot{X}_t\} - 2\,\{\ddot{X}_t\} \tag{4.60}$$

$$\{\ddot{X}_{t+\theta\Delta t}\} = \frac{3}{\theta\Delta t} (\{X_{t+\theta\Delta t}\} - \{X_t\}) - 2\,\{\dot{X}_t\} - \frac{\theta\Delta t}{2} \{\ddot{X}_t\} \tag{4.61}$$

We employ Eq. (4.13) at time $\tau + \Delta t$ to obtain a solution for displacement, velocity, and acceleration at time $t + \Delta t$. Since accelerations vary linearly, a linear projected force vector is used such that

$$[M]\,\{\ddot{X}_{t\,+\,\theta\Delta t}\} + [C]\,\{\dot{X}_{t+\theta\Delta t}\} + [K]\,\{X_{t+\theta\Delta t}\} = \{F_{t+\theta\Delta t}\} \tag{4.62}$$

where $\{F_{t+\theta\Delta t}\} = \{F_t\} + \theta\Delta t\,(\{F_{t+\theta\Delta t}\} - \{F_t\})$.

By substituting the expressions for $\{\ddot{X}_{t+\theta\Delta t}\}$ and $\{\dot{X}_{t+\theta\Delta t}\}$ from Eq. (4.60) and (4.61), respectively, into Eq.(4.62), we get

$$\lfloor \overline{M} \rfloor \{X_{t+\theta\Delta t}\} = \{\overline{F}_{t+\Delta t}\} \tag{4.63}$$

where the effective mass matrix $\lfloor \overline{M} \rfloor$ and the effective force vector $\{\overline{F}_{t+\Delta t}\}$ are given by

$$[\overline{M}] = \frac{6}{\theta^2 \Delta t^2}[M] + \frac{3}{\theta\Delta t}[C] + [K] \tag{4.64}$$

$$\{\overline{F}_{t+\theta\Delta t}\} = \{F_{t+\theta\Delta t}\} + (\frac{6}{\theta^2\Delta t^2}[M] + \frac{3}{\theta\Delta t}[C])\{X_t\}$$
$$+(\frac{6}{\theta\Delta t}[M] + 2[C])\{\dot{X}_t\} + (2[M] + \frac{\theta\Delta t}{2}[C])\{\ddot{X}_t\} \tag{4.65}$$

The solution of Eq. (4.63) gives $\{X_{t+\theta\Delta t}\}$ which is then substituted into the following relationships to obtain the displacements, velocities, and accelerations at time $t + \Delta t$.

$$\{\ddot{X}_{t+\Delta t}\} = \frac{6}{\theta^2\Delta t^2}(\{X_{t+\theta\Delta t}\} - \{X_t\}) - \frac{6}{\theta^2\Delta t}\{\dot{X}_t\} + (1 - \frac{3}{\theta})\{\ddot{X}_t\} \tag{4.66}$$

$$\{\dot{X}_{t+\Delta t}\} = \{\dot{X}_t\} + \frac{\Delta t}{2}(\{\ddot{X}_{t+\Delta t}\} - \{\ddot{X}_t\}) \tag{4.67}$$

$$\{X_{t+\Delta t}\} = \{X_t\} + \Delta t\,(\{\dot{X}_t\} + \frac{\Delta t^2}{6}\{\ddot{X}_{t+\Delta t}\} - 2\{\ddot{X}_t\}) \tag{4.68}$$

When $\theta = 1.0$, the method reduces to the linear acceleration scheme. The method is unconditionally stable for linear dynamic systems when $\theta \geq 1.37$, and a value of $\theta = 1.4$ is often used for nonlinear dynamic systems. It may also be noted that no special starting procedures are needed, since $X, \dot{X}$ and $\ddot{X}$ are expressed at time $\Delta + \Delta t$ in terms of the same quantities at time t only. The complete algorithm used in the Wilson Theta method is given in Table 4.4.

Table 4.4

Algorithm based on Wilson Theta method

(*a*) Initial Computations:

1. Form stiffness $[K]$, mass $[M]$ and damping $[C]$ matrices
2. Initialize $\{X_0\}\{\dot{X}_0\}$ and $\{\ddot{X}_0\}$.
3. Select time step Δt and calculate integration constants, $\theta = 1.4$ (say):

$$a_0 = \frac{6}{(\theta\Delta t)^2};\ a_1 = \frac{2}{\theta\Delta t};\ a_2 = 2a_1;\ a_3 = \frac{\theta\Delta t}{2};\ a_4 = \frac{a_0}{\theta};\ a_5 = \frac{-a_2}{\theta};$$

$$a_6 = 1 - \frac{3}{\theta};\ a_7 = \frac{\Delta t}{2};\ a_8 = \frac{\Delta t^2}{6}$$

4. Form effective stiffness matrix: $\lfloor \overline{K} \rfloor = [K] + a_0\,[M] + a_1\,[C]$
5. Triangularize $[\overline{K}]$: $[\overline{K}] = [L][D][L]^T$

(*b*) For each time step:

1. Calculate effective force vector at time $t + \Delta t$:

$$\{\overline{F_{t+\theta\Delta t}}\} = \{F_t\} + \theta\,(\{F_{t+\Delta t}\} - \{F_t\}) + [M]\,(a_0\,\{X_t\} + a_2\,\{\dot{X}_t\} + 2\,\{\ddot{X}_t\})$$
$$+ [C]\,(a_1\,\{X_1\} + 2\,\{\dot{X}_t\} + a_3\,\{\ddot{X}_t\})$$

2. Solve for displacements at time $t + \theta\,\Delta t$:

$$\lfloor \overline{K} \rfloor \{X_{t+\theta\Delta t}\} = \{\overline{F_{t+\theta\Delta t}}\}$$

3. Calculated $\{X\}, \{\dot{X}\}$ and $\{\ddot{X}\}$ at time $t + \Delta t$:

$$\{\ddot{X}_{t+\Delta t}\} = a_4\,(\{X_{t+\theta\Delta t}\} - \{X_t\}) + a_5\,\{\dot{X}_t\} + a_6\,\{\ddot{X}_t\}$$
$$\{\dot{X}_{t+\Delta t}\} = \{\dot{X}_t\} + a_7\,(\{\ddot{X}_{t+\Delta t}\} + \{\ddot{X}_t\})$$
$$\{X_{t+\Delta t}\} = \{X_t\} + \Delta t\,\{\dot{X}_t\} + a_8\,(\{\ddot{X}_{t+\Delta t}\} + 2\,\{\ddot{X}_t\})$$

4.5.3 Newmark Beta Method

The Newmark Beta integration method is also based on the assumption that the acceleration varies linearly between two instants of time. Two parameters α and β are used in this method, which can be changed to suit the requirements of a particular problem. The expressions for velocity and displacements are given by

$$\dot{X}_{t+\Delta t} = \dot{X}_t + [(1-\alpha)\ddot{X}_t + a\ddot{X}_{t+\Delta t}]\,\Delta t \quad (4.69)$$

$$X_{t+\Delta t} = X_t + \dot{X}_t\,\Delta t + \left[(\frac{1}{2} - \beta)\ddot{X}_t + \beta\ddot{X}_{t+\Delta t}\right]\Delta t^2 \quad (4.70)$$

The parameters α and β indicate how much the acceleration enters into the velocity and displacement equations at the end of the interval Δt. Therefore, α and β are chosen to obtain the desired integration accuracy and stability. When α =1/6 and β =1/2, Eqs. (4.69) and (4.70) correspond to the linear acceleration method (which can also be obtained using $\theta = 1$ in Wilson method). When $\alpha = 1/2$ and $\beta = 0$, the acceleration is constant and equal to $\ddot{X}_t$ during each time interval Δt. If α =1/2 and β = 1/8, the acceleration is constant from the beginning as $\ddot{X}_t$ and then changes to $\ddot{X}_{t+\Delta t}$ in the middle of the time interval Δt. When α =1/2 and β = 1/4, this corresponds to the assumption that the acceleration remains constant at an average value of $(\ddot{X}_t + \ddot{X}_{t+\Delta t})/2$. The finite difference formulas for the Newmark Beta scheme are

$$\{\ddot{X}_{t+\Delta t}\} = \frac{1}{\beta\Delta t^2}\,(\{X_{t+\Delta t}\} - \{X_t\}) - \frac{1}{\beta\Delta t}\,\{\dot{X}_t\} - (\frac{1}{2\beta} - 1)\,\{\ddot{X}_t\} \quad (4.71)$$

$$\{\dot{X}_{t+\Delta t}\} = \frac{1}{\beta\Delta t}\,(\{X_{t+\Delta t}\} - \{X_t\}) - \left(\frac{\alpha}{\beta} - 1\right)\{\dot{X}_t\} - \Delta t\left(\frac{\alpha}{2\beta} - 1\right)\{\ddot{X}_t\} \quad (4.72)$$

Equation (4.13) can be employed to obtain a solution for displacements, velocity and accelerations at time $t + \Delta t$. Therefore, by substituting the expressions for $\{\ddot{X}_{t+\Delta t}\}$ and $\{\dot{X}_{t+\Delta t}\}$ from Eqs. (4.71) and (4.72), respectively, into Eq. (4.13), we get

$$\lfloor \overline{M} \rfloor \{X_{t+\Delta t}\} = \{\overline{F}_{t+\Delta t}\} \tag{4.73}$$

where the effective mass matrix $\lfloor \overline{M} \rfloor$ and the effective force vector $\{\overline{F}_{t+\Delta t}\}$ are given by

$$[\overline{M}] = \frac{1}{\beta \Delta t^2} [M] + \frac{\alpha}{\beta \Delta t} [C] + [K] \tag{4.74}$$

$$\begin{aligned} \{\overline{F}_{t+\Delta t}\} = \{F_{t+\Delta t}\} &+ \left[(\frac{1}{2\beta} - 1) [M] + \Delta t \, (\frac{\alpha}{2\beta} - 1) [C] \right] \{\ddot{X}_t\} \\ &+ \left[\frac{1}{\beta \Delta t} [M] + (\frac{\alpha}{\beta} - 1) [C] \right] \{\dot{X}_t\} \\ &+ \left[\frac{1}{\beta \Delta t^2} [M] + \frac{\alpha}{\beta \Delta t} [C] \right] \{X_t\} \end{aligned} \tag{4.75}$$

Solution of Eq. (4.73) gives $\{X_{t+\Delta t}\}$, which is then substituted into (4.71) and (4.72) in order to obtain the accelerations and velocities at time $t + \Delta t$. One of the features of Newmark Beta method is that for linear systems the amplitude is conserved and the response is unconditionally stable, provided that $\alpha \geq \frac{1}{2}$ and $\beta \geq \frac{1}{4} (\alpha + \frac{1}{2})^2$. For values of $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{4}$ the largest truncation errors occur in the frequency of the response as opposed to other β values. It is also important to note that unless $\beta = \frac{1}{2}$, there is a spurious damping introduced, proportional to $(\beta - \frac{1}{2})$. If $\beta = 0$, a negative damping results; this involves a self-excited vibration arising solely from the numerical procedure. In a likewise manner, if β is greater than $\frac{1}{2}$, a positive damping is introduced which reduces the magnitude of the response even without real damping in the problem. For a multidegree of freedom system in which a number of modes constitute the total response, the peak amplitude may not be correct. The complete algorithm using the Newmark Beta integration method is given in Table 4.5.

Table 4.5

Algorithm based on Newmark Beta method

(*a*) Initial Computations:

1. Form stiffness $[K]$, mass $[M]$ and damping $[C]$ matrices
2. Initialize $\{X_0\}, \{\dot{X}_0\}$ and $\{\ddot{X}_0\}$.
3. Select time step Δt, parameters α and β, and calculate integration constants,

   $\beta \geq 0.5$; $\alpha \geq 0.25(0.5 + \beta)^2$

$$a_0 = \frac{1}{\beta(\Delta t)^2};\ a_1 = \frac{\alpha}{\beta \Delta t};\ a_2 = \frac{1}{\beta \Delta t};\ a_3 = \frac{1}{2\beta};\ a_3 = \frac{1}{2\beta} - 1;\ a_4 = \frac{\alpha}{\beta} - 1;$$

$$a_5 = \frac{\Delta t}{2}\left(\frac{\alpha}{\beta} - 2\right);\ a_6 = \Delta t\,(1 - \beta);\ a_7 = \beta \Delta t$$

4. Form effective stiffness matrix:

$$\lfloor \overline{K} \rfloor = [K] + a_0\,[M] + a_1\,[C]$$

5. Triangularize $[\overline{K}]: [\overline{K}] = [L]\,[D]\,[L]^T$

(*b*) For each time step:

1. Calculate effective force vector at time $t + \Delta t$:

$$\{\overline{F}_{t+\Delta t}\} = \{F_{t+\Delta t}\} + [M]\,(a_0\,\{X_t\} + a_2\,\{\dot{X}_t\} + a_2\,\{\dot{X}_t\} + a_3\,\{\ddot{X}_t\})$$
$$+ [C]\,(a_1\,\{X_t\} + a_4\,\{\dot{X}_t\} + a_5\,\{\ddot{X}_t\})$$

2. Solve for displacements at time $t + \Delta t$

$$\lfloor \overline{K} \rfloor \{X_{t+\Delta t}\} = \{\overline{F}_{t+\Delta t}\}$$

3. Calculate $\{\dot{X}\}$ and $\{\ddot{X}\}$ at time $t + \Delta t$:

$$\{\ddot{X}_{t+\Delta t}\} = a_0\,(\{X_{t+\Delta t}\} - \{X_t\}) - a_2\,\{\dot{X}_t\} - a_3\,\{\ddot{X}_t\}$$

$$\{\dot{X}_{t+\Delta t}\} = a_1\,(\{X_{t+\Delta t}\} - \{X_t\} - a_4\,\{\dot{X}_t\} - a_5\,\{\ddot{X}_t\}$$

4.5.4 Park Stiffly Stable Method

The Park Stiffly Stable method is an accurate method for low frequency ranges and stable for all higher-frequency components. Using a linear combination of the following two difference formulas derives the velocity in the Park Stiffly Stable method:

$$\dot{X}_{t+\Delta t} = \frac{1}{6\Delta t}\,[11X_{t+\Delta t} - 18X_t + 9X_{t-\Delta t} - 2X_{t-2\Delta t}] \tag{4.76}$$

$$\dot{X}_{t+\Delta t} = \frac{1}{2\Delta t}\,[2X_{t+\Delta t} - 4X_t + X_{t-\Delta t}] \tag{4.77}$$

The linear combination of (4.76) and (4.77) gives

$$\dot{X}_{t+\Delta t} = \frac{1}{4\Delta t}\,[3X_{t+\Delta t} - 4X_t + X_{t-\Delta t})$$
$$+ \frac{1}{12\Delta t}\,(11X_{t+\Delta t} - 18X_t + 9X_{t-\Delta t} - 2X_{t-2\Delta t}) \tag{4.78}$$

or

$$\dot{X}_{t+\Delta t} = \frac{1}{6\Delta t}\,(10X_{t+\Delta t} - 15X_t + 6X_{t-\Delta t} - X_{t-2\Delta t}) \tag{4.79}$$

Similarly, for the acceleration we obtain

$$\ddot{X}_{t+\Delta t} = \frac{1}{6\Delta t}\,(10\dot{X}_{t+\Delta t} - 15\dot{X}_t + 6\dot{X}_{t-\Delta t} - \dot{X}_{t-2\Delta t}) \tag{4.80}$$

The difference formulas in the Park Stiffly method are given by

$$\{\ddot{X}_{t+\Delta t}\} = \frac{1}{6\Delta t}\,[10\,\{\dot{X}_{t+\Delta t}\} - 15\,\{\dot{X}_t\} + 6\{\dot{X}_{t-\Delta t}\} - \{\dot{X}_{t-2\Delta t}\}] \qquad (4.81)$$

$$\{\dot{X}_{t+\Delta t}\} = \frac{1}{6\Delta t}\,[10\{X_{t+\Delta t}\} - 15\{X_t\} + 6\{X_{t-\Delta t}\} - \{X_{t-2\Delta t}\}] \qquad (4.82)$$

We consider Eq. (4.13) to obtain solution for the displacements, velocities and accelerations at time $t + \Delta t$. By substituting the expressions for $\{\ddot{X}_{t+\Delta t}\}$ and $\{\dot{X}_{t+\Delta t}\}$ from (4.81) and (4.82), respectively, into (4.13), we get

$$\lfloor \overline{M} \rfloor \{X_{t+\Delta t}\} = \{\overline{F}_{t+\Delta t}\} \qquad (4.83)$$

where the effective mass matrix $\lfloor \overline{M} \rfloor$ and the effective force vector $\{\overline{F}_{t+\Delta t}\}$ are given by

$$[\overline{M}] = \frac{100}{36\Delta t^2}\,[M] - \frac{10}{6\Delta t}\,[C] + [K] \qquad (4.84)$$

$$\begin{aligned}\{\overline{F}_{t+\Delta t}\} &= \frac{15}{6\Delta t}\,[M]\,\{\dot{X}_t\} - \frac{1}{\Delta t}\,[M]\,\{\dot{X}_{t-\Delta t}\} + \frac{1}{6\Delta t}\,[M]\,\{\dot{X}_{t-2\Delta t}\} \\ &\quad + (\frac{150}{36\Delta t^2}\,[M] + \frac{15}{6\Delta t}\,[C])\,\{X_t\} - (\frac{10}{6\Delta t^2}\,[M] + \frac{1}{\Delta t}\,[C])\,\{X_{t-\Delta t}\} \\ &\quad + (\frac{1}{36\Delta t^2}\,[M] + \frac{1}{6\Delta t}\,[C])\,\{X_{t-2\Delta t}\}\end{aligned} \qquad (4.85)$$

The solution of Eq.(4.83) gives $\{X_{t+\Delta t}\}$, which is then substituted in Eq.(4.82) to obtain velocities. The values of $\{\ddot{X}_{t+\Delta t}\}$ are then obtained by the use of Eq.(4.81). Note that in the Park Stiffly stable method, the calculation of $\{X_{t+\Delta t}\}$ requires the displacements and velocities at t, $t - \Delta t$ and $t - 2\Delta t$. Therefore, in order to obtain the solution at time Δt and $2\Delta t$, a special starting procedure is needed, which makes the method non-self-starting. The complete algorithm based on Park Stiffly stable method used in the integration is given in Table 4.6. The method requires a large computer memory in order to store the displacements and velocities for the two previous time steps.

Table 4.6

Algorithm based on Park Stiffly stable method

(*a*) Initial Computations:

1. Form stiffness $[K]$, mass $[M]$, and damping $[C]$ matrices
2. Initialize $\{X_0\}, \{\dot{X}_0\}$, and $\{\ddot{X}_0\}$.
3. Select time step Δt and calculate integration constants:

$$a_0 = \frac{10}{6\Delta t};\, a_1 = \frac{-15}{6\Delta t};\, a_2 = \frac{1}{\Delta t};\, a_3 = \frac{-1}{6\Delta t}$$

4. Form effective stiffness matrix:

$$\lfloor \overline{K} \rfloor = a_0^2\,[M] - a_0\,[C] + [K]$$

5. Triangularize $[\overline{K}]$: $[\overline{K}] = [L]\,[D]\,[L]^T$

(*b*) For each time step:

1. Calculate effective force vector at time $t + \Delta t$:

$$\{\overline{F}_{t+\Delta t}\} = (-a_1\{\dot{X}_t\} - a_2\{\dot{X}_{t-\Delta t}\} - a_3\{\dot{X}_{t-2\Delta t}\} - a_0 a_1\{X_t\} - a_0 a_2\{X_{t-\Delta t}\} + a_3^2\{X_{t-2\Delta t}\})[M] - (a_1\{X_t\} + a_2\{X_{t-\Delta t}\} + a_3\{X_{t-2\Delta t}\})[C]$$

2. Solve for displacements at time $t + \Delta t$

$$\lfloor \overline{M} \rfloor \{X_{t+\Delta t}\} = \{\overline{F}_{t+\Delta t}\}$$

3. Calculate $\{\dot{X}\}$ and $\{\ddot{X}\}$ at time $t + \Delta t$:

$$\{\dot{X}_{t+\Delta t}\} = a_0\{X_{t+\Delta t}\} + a_1\{X_t\} + a_2\{X_{t-\Delta t}\} + a_3\{X_{t-2\Delta t}\}$$

$$\{\ddot{X}_{t+\Delta t}\} = a_0\{\dot{X}_{t+\Delta t}\} + a_1\{\dot{X}_t\} + a_2\{\dot{X}_{t-\Delta t}\} + a_3\{\dot{X}_{t-2\Delta t}\}$$

4.6 SUMMARY

In this chapter we have briefly reviewed the direct numerical integration methods for the solution of a single or system of differential equations. Many numerical methods are available for the solutions of the response of dynamic systems. We have discussed several widely used step-by-step numerical integration methods for linear dynamic response analysis. A brief description of these integration methods is presented and their application is illustrated. The integration schemes considered were three explicit and four implicit methods. They are the explicit schemes (the central difference method, two-cycle interaction with trapezoidal rule and fourth order Runge-Kutta method) and the implicit schemes(Houbolt method, Wilson Theta method, Newmark Beta method and the Park Stiffly stable method). Application of these direct numerical integration methods is illustrated with numerical examples for a linear dynamic system.

The use of a particular integration method is mainly dependent on the nature of the problem and is often dictated by the desired solution accuracy.

4.7 EXAMPLE PROBLEMS AND SOLUTIONS USING MATLAB

Example 4.1. *Find the response of a viscously damped single degree of freedom system subjected to a force*

$$F(t) = F_0\left(1 - \sin\frac{\pi t}{2t_0}\right)$$

with the following data: $F_0 = 2N$, $t_0 = \pi$ *seconds,* $m = 2$ *kg,* $c = 0.3$ *Ns/m and* $k = 1$ *N/m. The values of the displacement and velocity of the mass at* $t = 0$ *are zero. Use the central difference method. Choose* $\Delta t = 1$, 0.1 *and* 0.5 *seconds and compare the results.*

Solution:

This is a single-degree of freedom system problem with all initial conditions zero. The following MATLAB program is executed to obtain the results.

```
% INITIAL VALUES
```